Quantization of Point-Like Particles and Consistent Relativistic Quantum Mechanics

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Abstract

We revise the problem of the quantization of relativistic particle models (spinless and spinning), presenting a modified consistent canonical scheme. One of the main point of the modification is related to a principally new realization of the Hilbert space. It allows one not only to include arbitrary backgrounds in the consideration but to get in course of the quantization a consistent relativistic quantum mechanics, which reproduces literally the behavior of the one-particle sector of the corresponding quantum field. In particular, in a physical sector of the Hilbert space a complete positive spectrum of energies of relativistic particles and antiparticles is reproduced, and all state vectors have only positive norms.

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I. INTRODUCTION

Already for a long time there exists a definite interest in studying (construction and quantization) of classical and pseudoclassical models of relativistic particles (RP) of different kinds. There are various reasons for that. One may mention, for example, a widely spread explanation that RP is a prototype string and with that simple example one can study many of the problems related to string quantization. However, we believe that a more profound motivation is stipulated by a desire to understand better basic principles of the quantization theory and to prove that there exist a consistent classical and quantum descriptions at least for noninteracting (between each other) RP of different kinds (with different masses, spins, and in different space-time dimensions), moving in external backgrounds. The problem may be considered as a supplementary one to the problem of relativistic wave equation construction for the particles of different kinds. Indeed, quantizing a classical or pseudoclassical model of RP we aspire to reproduce a quantum mechanics, which in a sense is based on the corresponding relativistic wave equation. And here it is necessary to formulate more precisely the aim of the quantization problem. Indeed, there is a common opinion that the construction of a consistent relativistic quantum mechanics on the base of the relativistic wave equations meets well-known difficulties related to the existence of infinite number of negative energy levels (energy levels, which correspond to antiparticles appear with negative sign in the spectrum), to the existence of negative vector norms, and difficulties related to localized state construction (position operator problem), part of which may be only solved in the second-quantized theory, see for example [1–6]. In this relation one ought to mention some attempts to to construct relativistic wave equations for the wave functions, which realize infinite-dimensional representations of the Lorentz group [7,8]. Thus, the quantization problem under consideration may be formulated with different degrees of claim. The simplest and most widely used approach is to apply some convenient in the concrete case (but not always the most convinced and well-grounded) scheme of quantization in the given case to arrive in a way to a corresponding relativistic wave equation, without any attempt

to demonstrate that a consistent quantum mechanics was constructed, due to the above mentioned belief that it cannot be done. To our mind the aim has to be more ambitious: namely, in the course of a first quantization of a RP model one has to try to construct a relativistic quantum mechanics consistent to the same extent to which a one-particle description is possible in the frame of the corresponding quantum field theory (in the frame of the second quantized theory). We will demonstrate in the present article a possible way of realizing this with the example of the quantization problem of a spinless and spinning charged RP moving in arbitrary external electromagnetic and gravitational fields.

First of all we have to define more precisely what we mean by convincing and well-grounded quantization. To our mind it has to be a consistent general scheme, but not some leading considerations, which allow one to predict in a way some basic aspects of a corresponding quantum theory of the classical model under consideration. For a consistent scheme of this nature one may refer, for example, to the canonical quantization of gauge theories, in which the physical sector may be selected already on the classical level by means of a gauge fixing, and the state space may be constructed and analyzed in detail. That may be also any equivalent to the canonical quantization scheme, which allows one to achieve the same final result. An alternative and frequently used method of Dirac quantization, in which the gauge conditions are not applied on the classical level, and first-class constraints are used as operators to select the physical sector in the state space, contains some essential intrinsic contradictions, in particular, one cannot formulate a consistent prescription to construct the appropriate Hilbert space in this case. Besides, there is no general proof of the equivalence of this method to the canonical quantization. All that does not allow one to consider this quantization scheme as a consistent one in the above mentioned sense. One ought to say that this method is rather popular due to its simplicity and due to the possibility to sometimes quickly reach a desired result. In particular, from the point of view of this method the problem of quantization, for example of scalar RP looks, in a sense, trivial. Indeed, the first-class constraint $p^2 = m^2$ reproduces in this scheme immediately something which looks like Klein-Gordon equation. In spite of the fact that still nothing

has been said how a consistent quantum mechanics may appear starting with that point, sometimes they accept it as a final solution of the problem. In connection with this we would like to repeat once again that the problem, as we see it, is not to "derive" in any way the Klein-Gordon or Dirac equations. The question is: May or not a consistent relativistic quantum mechanics be reproduced in the course of honest application of a well developed general scheme of a quantization, to some classical or pseudoclassical models of RP? Under the consistent relativistic quantum mechanics we mean a reduction of the quantum field theory of a corresponding field (scalar, spinor, etc.) to the one particle sector, if such a reduction may be done. The latter is possible if the interaction of the given quantum field with external backgrounds does not lead to a particle creation.

What was done before to solve such formulated problem? Which kind of difficulties one meets here, and how the present work may contribute to progress in this direction?

Usually the above mentioned models of RP are formulated in covariant and reparametrization invariant form. Due to the latter invariance, which is, in fact, a gauge invariance, one meets here all the problems related to the quantization of such systems, e.g. zero-Hamiltonian phenomenon and the time problem, which are crucial, for example, also for the quantization of such important reparametrization invariant theory as general relativity. Besides, the problem of spinning degree of freedom description turns out to be nontrivial in RP models. Here there are two competing approaches, one which uses Grassmann variables for spin description, and gives rise to the pseudoclassical mechanics, and another one, which uses variables from a compact bosonic manifold. Both approaches have their own problems related, in particular, to higher spin description and introduction for such spins an interaction to external backgrounds. We do not touch here the problem of path integral quantization of relativistic particles. Readers interested in that question may look, for example, up the articles [9].

One of possible approach to the canonical quantization of the relativistic particle (spinless and spinning) was presented in the papers [10,11] on the base of a special gauge, which fixes the reparametrization gauge freedom. It was shown how the Klein Gordon and Dirac equa-

tions appear in the course of the quantization from the corresponding Schrödinger equations. However, only a restricted class of external backgrounds (namely, constant magnetic field) was considered. One may see that the above quantum theory does not obey all symmetries of the corresponding classical model. Besides, an analysis of the equivalence between the quantum mechanics constructed in course of the quantization and the one-particle sector of the corresponding field theory was not done in all details. Moreover, from the point of view of the results of the present work, one can see that such an equivalence is not complete. Thus, the question: whether a consistent quantum mechanics in the above mentioned sense is constructed, remain. Attempts to generalize the consideration to arbitrary external electromagnetic background [12] have met some difficulties (even Klein-Gordon and Dirac equations were not reproduced in course of the quantization), which look even more complicated in the case of a RP moving in curved space-time (in an external gravitational field). As to the latter problem, it is enough to mention that even more simple corresponding nonrelativistic problem (canonical quantization of a particle in curved three-dimensional space, which has a long story [13]) attracts attention to the present day and shows different points of view on its solution [13–16]. The relativistic problem, which naturally absorbs all known difficulties of its nonrelativistic analog, is essentially more rich and complicated due to its gauge nature (reparametrization invariance). If the external gravitational field is arbitrary, then the problem can not be solved (even in the restricted sense to reproduce only the Klein-Gordon and Dirac equations) by complete analogy with the flat space case in an external constant magnetic field [11], how it was done in [17] for the static space-time. However, namely the general case is interesting from the principle point of view. It turns out that the whole scheme of quantization, which was used in [10,11] and repeated then in numerous works, has to be changed essentially to make it possible to include arbitrary external backgrounds (electromagnetic or gravitational) in the consideration and maintain all classical symmetries on the quantum level. Such a modified scheme of the canonical quantization of RP is described in the present article first in detail on the example of a spinless charged particle moving in arbitrary external electromagnetic and gravitational fields, and then it is

applied already briefly to the spinning particle case.

One of the main point of the modification is related to a principally new realization of the Hilbert space. It has allowed one not only to include arbitrary backgrounds in the consideration but to solve the problem completely, namely to get in course of the quantization the consistent relativistic quantum mechanics, which reproduces literally the behavior of the one-particle sector of the quantum field (in external backgrounds, which do not create particles from vacuum). In particular, in a physical sector of the Hilbert space complete positive spectrum of energies of relativistic particles and antiparticles is reproduced, and corresponding state vectors have only positive norms.

The article is organized in the following way: In Sect.II we present a detailed Hamiltonian analysis of the theory of a classical relativistic particle with a reparametrization invariant action in external electromagnetic and gravitational backgrounds. We focus our attention on the selection of physical degrees of freedom and on the adequate gauge fixing. Due to the fact that after our gauge fixing we remain with time dependent constraint system, a method of treatment and quantization of such systems is briefly discussed in the end of the Section. In Sect.III we proceed with canonical first quantization procedure. Here we discuss in details Hilbert space construction, realization of all physical operators in this space, and in the end of the section we reformulate the evolution of the system under consideration in terms of a physical time. In Sect.IV we demonstrate a full equivalence of the quantum mechanics constructed to the dynamics of one-particle sector of the corresponding field theory in backgrounds, which do not create particles from vacuum. In the Sect. V we generalize consideration to the spinning particle case. To make the consideration complete we present also Dirac quantization scheme both in scalar and spinning case. Treating the spinless case we use widely results of brief consideration of the quantum field theory of scalar field in the electromagnetic and gravitational backgrounds, which are presented in the Appendix to the article.

II. CLASSICAL SPINLESS RELATIVISTIC PARTICLE

The classical theory of a relativistic charged spinless particle placed in $3+1$-dimensional Riemannian space-time (with coordinates $x=(x^\mu)=(x^0,x^i)=(x^0,\mathbf{x})$, and a metric tensor $g_{\mu\nu}(x)$, $\mu=0,1,2,3$, $\; i=1,2,3)$, and interacting with an external electromagnetic field, may be described by a reparametrization invariant action

$$S=\int_0^1 Ld\tau\ ,\quad L=-m\sqrt{\dot{x}^\mu g_{\mu\nu}(x)\dot{x}^\nu}-q\dot{x}^\mu g_{\mu\nu}(x)A^\nu(x)=-m\sqrt{\dot{x}^2}-q\dot{x}A\ , \tag{2.1}$$

where $\dot{x}^\mu=dx^\mu/d\tau$; τ is a real evolution parameter, which plays the role of time in the problem under consideration; q is an algebraic charge of the particle; and $A^\mu(x)$ are potentials of an external electromagnetic field. The action (2.1) is invariant under reparametrizations $x^m(\tau)\to x'^\mu(\tau)=x^m(f(\tau))$, where $f(\tau)$ is an arbitrary function subjected only the following conditions: $\dot{f}(\tau)>0$, $f(0)=0$, $f(1)=1$. The reparametrizations may be interpreted as gauge transformations whose infinitesimal form is $\delta x^\mu(\tau)=\dot{x}^\mu(\tau)\epsilon(\tau)$, where $\epsilon(\tau)$ is τ-dependent gauge parameter.

For the purposes of the quantization it is preferable to select a reference frame, which admits a time synchronization over all space. Such a reference frame corresponds to a special gauge $g_{0i}=0$ of the metric[1]. It is called (at $g_{00}=1$) synchronous reference frame according to [18], or corresponds to Gaussian coordinates according to [19]. Such a reference frame exists always for any real space-time.

Our aim is the canonical quantization, thus, we need first a detailed Hamiltonian analysis of the problem. Let us denote via p_μ canonical momenta conjugated to the coordinates x^μ,

$$p_\mu=\frac{\partial L}{\partial\dot{x}^\mu}=-\frac{mg_{\mu\nu}\dot{x}^\nu}{\sqrt{\dot{x}^2}}-qA_\mu\ ,\quad A_\mu=g_{\mu\nu}A^\nu\ . \tag{2.2}$$

Let us introduce an important for the further consideration discrete quantity $\zeta=\pm1$, which is defined in the phase space,

[1] In such a gauge $g^{00}=g_{00}^{-1}$, $g^{ik}g_{kj}=\delta^i_j$.

$$\zeta = -\text{sign}\left[p_0 + qA_0\right] \ . \tag{2.3}$$

Due to the fact that $g_{00} > 0$ an important relation follows from the Eq. (2.2) at $\mu = 0$,

$$\text{sign}(\dot{x}^0) = \zeta \ . \tag{2.4}$$

It follows also from (2.2) that there exists a primary constraint

$$\Phi_1' = \left[p_\mu + qA_\mu\right] g^{\mu\nu} \left[p_\nu + qA_\nu\right] - m^2 = 0 \ , \ \ g^{\alpha\nu} g_{\nu\beta} = \delta^\alpha_\beta \ . \tag{2.5}$$

On the other hand, it is clear that the relation (2.5) is, in fact, a constraint on the modulus of $p_0 + qA_0$ only,

$$|p_0 + qA_0| = \omega \ , \ \ \omega = \sqrt{g_{00}\left\{m^2 - \left[p_k + qA_k\right] g^{kj} \left[p_j + qA_j\right]\right\}} \ . \tag{2.6}$$

Taking into account (2.3), we may write an equivalent to (2.5) constraint in the following linearized in p_0 form

$$\Phi_1 = p_0 + qA_0 + \zeta\omega = 0 \ . \tag{2.7}$$

Indeed, it is easy to see that Φ_1' is a combination of the constraint (2.7), $\Phi_1' = g^{00}\left[-2\zeta\omega\Phi_1 + (\Phi_1)^2\right]$. Further we are going to work with the constraint Φ_1, in particular, one see explicitly that it imposes no restrictions on ζ. That is especially important for our consideration.

To construct the total Hamiltonian in a theory with constraints we have to identify the primary-expressible velocities and primary-inexpressible ones [20]. The role of the former velocities are playing here $\text{sign}(\dot{x}^0)$ and $\dot{x}^i$. Indeed, the first quantity is expressed via the phase space variables (see (2.4)). Besides, it follows from the equation (2.2) that

$$\sqrt{\dot{x}^2} = m g_{00}\omega^{-1}\lambda \ , \ \ \dot{x}^i = -g_{00}\omega^{-1}\left[p_k + qA_k\right]\lambda g^{ki} \ , \ \ \lambda = |\dot{x}^0| \ . \tag{2.8}$$

Thus, we may regard $\dot{x}^i$ as primary-expressible velocity as well, and λ as primary-inexpressible velocity. We may expect that the latter quantity will appear as a Lagrange

multiplier in the total Hamiltonian $H^{(1)}$ of the theory. Indeed, constructing such a Hamiltonian according to the standard procedure [20–22], we get

$$H^{(1)} = (p_\mu \dot{x}^\mu - L)|_{\dot{x}^\mu = f(x,p,\lambda)} = \zeta \lambda \Phi_1 \, . \tag{2.9}$$

It vanishes on the constraint surface in accordance with the reparametrization invariance nature of the formulation [22–25]. One can make sure that the equations

$$\dot{\eta} = \{\eta, H^{(1)}\}, \;\; \Phi_1 = 0, \;\; \lambda > 0, \;\; \eta = (x^\mu, p_\mu) \, , \tag{2.10}$$

are equivalent to the Lagrangian equations of motion[2] (with account taken of the definition of the momenta (2.2)). Equations (2.10) are equations of motion of a Hamiltonian theory with a primary constraint (2.7). In these equations λ is an undetermined Lagrange multiplier, about which part of the information ($\lambda > 0$) is already available (the latter condition is necessary to provide the above mentioned equivalence).

The consistency condition ($\dot{\Phi}_1 = 0$) for the constraint (2.7) does not lead to any new secondary constraints and λ is no longer defined. Thus, (2.7) is a first-class constraint. What kind of gauge fixation one can chose to transform the theory to second-class constraint type? Let us consider, for example, the case of a neutral ($q = 0$) particle. In this case the action (2.1) is invariant under the time inversion $\tau \to -\tau$. Since the gauge symmetry in the case under consideration is related to the invariance of the action under the changes of the variables τ, there appear two possibilities: namely, to include or not to include the above discrete symmetry in the gauge group together with the continuous reparametrizations. Let us study the former possibility and include the time inversion in the gauge group. Then the gauge conditions have to fix the gauge freedom, which corresponds to both kind of

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[2]Here and in what follows the Poisson brackets are defined as [20]

$$\{\mathcal{F}, \mathcal{G}\} = \frac{\partial_r \mathcal{F}}{\partial q^a}\frac{\partial_l \mathcal{G}}{\partial p_a} - (-1)^{P_\mathcal{F} P_\mathcal{G}} \frac{\partial_r \mathcal{G}}{\partial q^a}\frac{\partial_l \mathcal{F}}{\partial p_a} \, ,$$

where $P_\mathcal{F}$ and $P_\mathcal{G}$ are Grassmann parities of $\mathcal{F}$ and $\mathcal{G}$ respectively.

symmetries, namely, to fix the variable $\lambda = |\dot{x}^0|$, which is related to the reparametrizations, and to fix the variable $\zeta = \ \mathrm{sign}\,\dot{x}^0$, which is related to the time inversion. To this end we may select a supplementary condition (the chronological gauge) of the form $\Phi_2 = x^0 - \tau = 0$. The consistency equation $\dot{\Phi}_2 = 0$ leads on the constraint surface to the relation $\partial_\tau \Phi_2 + \{\Phi_2, H^{(1)}\} = -1 + \lambda\zeta = 0$, which results in $\zeta\lambda = 1$. Remembering that $\lambda \geq 0$, we get $\zeta = 1,\ \lambda = 1$. Suppose we do not include the time inversion in the gauge group. That is especially natural when $q \neq 0$, $A_\mu \neq 0$, since in this case the time inversion is not anymore a symmetry of the action. Under the above supposition the above suplimentary condition is not anymore a gauge, it fixes not only the reparametrization gauge freedom (fixes λ) but it fixes also the variable ζ, which is now physical. A possible gauge condition, which fixes only λ, has the form [10,11]:

$$\Phi_2 = x^0 - \zeta\tau = 0 \, . \tag{2.11}$$

Indeed, the consistency condition $\dot{\Phi}_2 = 0$ leads to the equation $\partial_\tau \Phi_2 + \{\Phi_2, H^{(1)}\} = -\zeta + \lambda\zeta = 0$, which fixes $\lambda = 1$ and retains ζ as a physical variable. To make more clear the meaning of the discrete variable $\zeta = \pm 1$ let us study the equations of motion (2.10) in the gauge (2.11). Selecting for simplicity the flat space case ($g_{\mu\nu} = \eta_{\mu\nu} = \ \mathrm{diag}(1, -1, \ldots, -1)$), we can see that these equations may be written in the following form:

$$\begin{aligned} &\frac{d\mathcal{P}_i^{kin}}{d(\zeta\tau)} = (\zeta q)\left[F_{0i} + F_{ji}\frac{dx^j}{d(\zeta\tau)}\right] \, , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, , \\ &\frac{dx^i}{d(\zeta\tau)} = \frac{\mathcal{P}_i^{kin}}{\sqrt{m^2 + (\mathcal{P}^i_{kin})^2}} \, , \quad \frac{d\zeta}{d(\zeta\tau)} = 0 \, , \quad \mathcal{P}_i^{kin} = \zeta p_i + (\zeta q) A_i \, . \end{aligned} \tag{2.12}$$

It is natural now to interpret $\zeta\tau = x^0$ as a physical time[3], $\zeta p_i = \mathcal{P}_i$ as a physical momentum, and $\frac{dx^j}{d(\zeta\tau)} = \frac{dx^j}{dx^0} = v^j$ as a physical three-velocity. Then $\mathcal{P}_i^{kin} = \mathcal{P}_i + (\zeta q) A_i$ is the kinetic

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[3]In a sense we return into the consideration the initial variable x^0, which was gauged out by means of a gauge condition. However, now x^0 has the status of an evolution parameter but not a dynamical variable.

momentum of a particle with the charge ζq. In terms of such quantities the equations (2.12) take the form:

$$\frac{d\boldsymbol{\mathcal{P}}_{kin}}{dx^0} = (\zeta q)\left\{\mathbf{E} + [\mathbf{v},\mathbf{H}]\right\}\ ,\ \mathbf{v} = \frac{\boldsymbol{\mathcal{P}}_{kin}}{\sqrt{m^2 + \boldsymbol{\mathcal{P}}_{kin}^2}},\ \ \frac{d\zeta}{dx^0} = 0,\ \ \zeta = \pm 1\ , \tag{2.13}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic fields respectively and $\mathbf{v} = (v^i),\ \boldsymbol{\mathcal{P}}^{kin} = \left(\mathcal{P}_i^{kin}\right)$. Equations (2.13) are well recognized classical relativistic equations of motion for a charge ζq moving in an external electromagnetic field [18]. Now we may conclude that trajectories with $\zeta = +1$ correspond to the charge q, while those with $\zeta = -1$ correspond to the charge $-q$ (that was first pointed out in [10,11]). The sign of the charge ζ is a conserved quantity in the theory. This interpretation remains also valid in the presence of an arbitrary gravitational background. Thus, the theory with the action (2.1) describes states with both sign of the electric charge. This doubling of the state space on classical level (due to the existence of the variable ζ) naturally appears also on the quantum level, how it will be demonstrated below, and is decisive for the construction of a consistent relativistic quantum mechanics.

The set of the constraints $\Phi_a = 0,\ a = 1, 2$ is now second-class. However, it depends explicitly on the time (namely Φ_2 does). In this case an usual canonical quantization by means of Dirac brackets has to be modified (see for details [20]). In the case of a particle in a flat space, moving in an magnetic field with time independent potentials [10,11], or in completely similar case of a particle in static space-time [17], it is possible to make explicitly a simple canonical transformation, which transforms the constraint surface to a time independent form, and then proceed to the usual scheme of the canonical quantization by means of Dirac brackets. The above mentioned canonical transformation depends explicitly on time, thus, a new effective non-vanishing on the constraint surface Hamiltonian appears. In the case under consideration, with arbitrary gravitational and electromagnetic backgrounds, to find such a canonical transformation seems to be a difficult task. Nevertheless, the problem of the canonical quantization may be solved on the base of the approach to non-stationary second-class constraints developed in [20] (similar results were obtained by a geometrical approach in [26]). Below we present such an approach, which allows one to treat easily the

backgrounds of general form and at the same time clarifies some ambiguities hidden in the scheme of quantization of the original papers [10,11], which was used in many following publications devoted to the canonical quantization of the classical and pseudoclassical models (see [27] and Ref. therein). First, we recall briefly the treatment of Ref. [20] for systems with non-stationary second-class constraints.

Consider a theory with second-class constraints $\Phi_a(\eta,t)=0$ (where $\eta=(x^i,\pi_i)$ are canonical variables), which may explicitly depend on time t. Then the equations of motion for such a system may be written by means of the Dirac brackets, if one formally introduces a momentum ϵ conjugated to the time t, and defines the Poisson brackets in the extended phase space of canonical variables $(\eta;t,\epsilon)$,

$$\dot{\eta}=\{\eta,H+\epsilon\}_{D(\Phi)},\quad \Phi(\eta,t)=0\ , \tag{2.14}$$

where H is the Hamiltonian of the system, and $\{A,B\}_{D(\phi)}$ is the notation for the Dirac bracket with respect to the system of second–class constraints ϕ. The Poisson brackets, wherever encountered, are henceforth understood as ones in the above mentioned extended phase space. The quantization procedure in Heisenberg picture can be formulated in that case as follows. The variables η of the theory are assigned the operators $\hat{\eta}$, which satisfy the following equations and commutation relations[4]

$$\dot{\hat{\eta}}=\{\eta,H+\epsilon\}_{D(\Phi)}\big|_{\eta=\hat{\eta}}\ ,\quad [\hat{\eta},\hat{\eta}']=i\ \{\eta,\eta'\}_{D(\Phi)}\big|_{\eta=\hat{\eta}}\quad \Phi(\hat{\eta},t)=0\ . \tag{2.15}$$

The total time evolution is controlled only by the first set of the equations (2.15) since the state vectors do not depend on time in the Heisenberg picture. In the general case such an evolution is not unitary. Suppose, however, that a part of the set of second-class constraints consists of supplementary gauge conditions, the choice of which is in our hands. In this case

[4] In fact, the commutator here is understood as a generalized one, it is a commutator in case if one or both operators have Grassmann even parities, and it is an anticommutator if both operators have Grassmann odd parities.

one may try to select these gauge conditions in a special form to obtain an unitary evolution. The evolution is unitary if there exists an effective Hamiltonian $H_{eff}(\eta)$ in the initial phase space of the variables η so that the equations of motion (2.14) may be written as follows

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\Phi)} = \{\eta, H_{eff}\}_{D(\Phi)} \,, \quad \Phi(\eta, t) = 0 \,. \tag{2.16}$$

In this case, (due to the commutation relations (2.15)) the quantum operators $\hat{\eta}$ obey the equations (we disregard here problems connected with operator ordering)

$$\dot{\hat{\eta}} = -i[\hat{\eta}, \hat{H}] \,, \quad \hat{H} = H_{eff}(\hat{\eta}) \,, \quad [\hat{\eta}, \hat{\eta}'] = i \left.\{\eta, \eta'\}_{D(\Phi)}\right|_{\eta=\hat{\eta}} \,, \quad \Phi(\hat{\eta}, t) = 0 \,. \tag{2.17}$$

The latter allows one to introduce a Schrödinger picture, where operators do not depend on time, but the evolution is controlled by the Schrödinger equation with the Hamiltonian $\hat{H}$. We may call the gauge conditions, which imply the existence of the effective Hamiltonians, as *unitary gauges*. Remember that in the stationary constraint case all gauge conditions are unitary [20]. As it is known [20], the set of second-class constraints can always be solved explicitly with respect to part of the variables $\eta_* = \Psi(\eta^*)$, $\eta = (\eta_*, \eta^*)$, so that η_* and η^* are sets of pairs of canonically conjugated variables $\eta_* = (q_*, p_*)$, $\eta^* = (q^*, p^*)$. We may call η^* as independent variables and η_* as dependent ones. In fact $\eta_* - \Psi(\eta^*) = 0$ is an equivalent to $\phi(\eta) = 0$ set of second-class constraints. One can easily demonstrate that it is enough to verify the existence of the effective Hamiltonian (the validity of relation (2.16)) for the independent variables only. Then the evolution of the dependent variables, which is controlled by the constraint equations, is also unitary.

Returning to our concrete problem, we remark that in the case under consideration the Hamiltonian H in the equations (2.14) vanishes (total Hamiltonian vanishes on the constraint surface). Thus, these equations take the form

$$\dot{\eta} = \{\eta, \epsilon\}_{D(\Phi)} = -\{\eta, \Phi_a\} C_{ab}\, \partial_\tau \Phi_b, \quad \Phi_a = 0 \,, \tag{2.18}$$

were $\eta = (x^\mu, p_\nu)$, and $C_{ab}\{\Phi_b, \Phi_c\} = \delta_{ac}$. Calculating the matrices $\{\Phi_a, \Phi_b\}$ and C_{ab} on the constraint surface, we get $\{\Phi_a, \Phi_b\} = \text{antidiag}(-1, 1)$, $C_{ab} = \{\Phi_b, \Phi_a\}$. Let us work now

with independent variables, which are in the case under consideration $\boldsymbol{\eta} = (x^k, p_k, \zeta)$. It is easy to see that (2.18) imply the following equations for such variables:

$$\dot{\boldsymbol{\eta}} = \{\boldsymbol{\eta}, \mathcal{H}_{eff}\} \ , \quad \zeta = \pm 1 \ , \tag{2.19}$$

where the effective Hamiltonian H_{eff} reads:

$$\mathcal{H}_{eff} = [\zeta q A_0(x) + \omega]_{x^0 = \zeta\tau} \ . \tag{2.20}$$

It particular, it follows from (2.19) that $\dot{\zeta} = 0$. One can see that the equations (2.19) are ordinary Hamiltonian equations of motion without any constraints. Thus, formally, all the problems with zero-Hamiltonian phenomenon and time dependence of the constraints remain behind. In fact, we have demonstrated that in the gauge under consideration (2.11) the dynamics in the physical sector is unitary and the corresponding effective Hamiltonian has been constructed explicitly.

III. FIRST QUANTIZATION OF SPINLESS PARTICLE MODEL

Now, the problem of the canonical operator quantization of the initial gauge theory is reduced to the quantization of a non-constrained Hamiltonian theory with the equations (2.19). We assume also the operator $\hat{\zeta}$ to have the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory. The equal time commutation relations for the operators $\hat{X}^k, \hat{P}_k, \hat{\zeta}$, which correspond to the variables x^k, p_k, ζ, we define according to their Poisson brackets. Thus, nonzero commutators are

$$[\hat{X}^k, \hat{P}_j] = i\hbar\delta_j^k \, , \quad \left(\hat{\zeta}^2 = 1\right) \ . \tag{3.1}$$

We are going to present a realization of such an operator algebra in a Hilbert space and construct there a quantum Hamiltonian $\hat{H}$ according to the classical expression (2.20).

In the capacity of the above mentioned Hilbert space we select a space $\mathcal{R}$, whose elements $\boldsymbol{\Psi} \in \mathcal{R}$ are $\mathbf{x}$-dependent four-component columns

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_{+1}(\mathbf{x}) \\ \Psi_{-1}(\mathbf{x}) \end{pmatrix} , \quad \Psi_\zeta(\mathbf{x}) = \begin{pmatrix} \chi_\zeta(\mathbf{x}) \\ \varphi_\zeta(\mathbf{x}) \end{pmatrix} , \tag{3.2}$$

where $\Psi_\zeta(\mathbf{x}),\ \zeta = \pm 1$ are two component columns with χ and φ being $\mathbf{x}$-dependent functions. The inner product in $\mathcal{R}$ is defined as follows[5]:

$$(\mathbf{\Psi}, \mathbf{\Psi}') = \left(\Psi_{+1}, \Psi'_{+1}\right) + \left(\Psi'_{-1}, \Psi_{-1}\right) , \tag{3.3}$$

$$(\Psi, \Psi') = \int \overline{\Psi}(\mathbf{x})\Psi'(\mathbf{x})d\mathbf{x} = \int \left[\chi^*(\mathbf{x})\varphi'(\mathbf{x}) + \varphi^*(\mathbf{x})\chi'(\mathbf{x})\right] d\mathbf{x} , \ \overline{\Psi} = \Psi^+\sigma_1 . \tag{3.4}$$

Later on one can see that such a construction of the inner product provides its form invariance under general coordinate transformations.

We seek all the operators in the block-diagonal form[6],

$$\hat{\zeta} = \mathrm{bdiag}\,(I, -I) \ , \ \ \hat{X}^k = x^k\mathbf{I} \ , \quad \hat{P}_k = \hat{p}_k\mathbf{I} \ , \ \ \hat{p}_k = -i\hbar\partial_k \ , \tag{3.5}$$

where I and $\mathbf{I}$ are 2×2 and 4×4 unit matrices respectively. One can easily see that such defined operators obey the commutation relations (3.1) and are Hermitian with respect to the inner product (3.3). Evolution of state vectors with the time parameter τ is controlled by the Schrödinger equation with a quantum Hamiltonian $\hat{H}$. The latter may be constructed as a quantum operator in the Hilbert space $\mathcal{R}$ on the base of the correspondence principle starting with its classical analog, which is $\mathcal{H}_{eff}$ given by Eq. (2.20). However, on this way we

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[5]Here and in what follows we use standard σ-matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

[6]Here and in what follows we use the following notations

$$\mathrm{bdiag}\,(A,\ B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ,$$

where A and B are some matrices.

meet two kind of problems. First of all, one needs to define the square root in the expression (2.6) on the operator level. Then one has to solve an ordering problem, it appears due to a non-commutativity of operators, which have to be situated under the square root sign in (2.6). Below we are going to discuss both problems. It seams to be instructive to do that first for a free particle in a flat space-time, and then in general case (in the presence of both backgrounds, electromagnetic and gravitational).

For a free particle in a flat space-time $g_{\mu\nu} = \eta_{\mu\nu}$, $A_\mu = 0$. Then $\mathcal{H}_{eff} = \omega = \sqrt{m^2 + (p_k)^2}$. We construct the quantum Hamiltonian as $\hat{H} = \hat{\Omega}$, where $\hat{\Omega}$ is an operator related to the classical quantity ω. Such an operator we define as follows:

$$\hat{\Omega} = \mathrm{bdiag}\,(\hat{\omega}, \hat{\omega}) \ , \quad \hat{\omega} = \begin{pmatrix} 0 & m^2 + (\hat{p}_k)^2 \\ 1 & 0 \end{pmatrix} .$$

Thus defined operator $\hat{\Omega}$ is Hermitian with respect to the inner product (3.3), its square $\hat{\Omega}^2 = [m^2 + (\hat{p}_k)^2]\,\mathbf{I}$, corresponds to the square of the classical quantity ω, and it is a well defined (in the space $\mathcal{R}$) operator function on the basic canonical operators $\hat{p}_k$. Thus, the square root problem is solved here due to the state space doubling (3.2). In the case under consideration we do not meet an ordering problem.

In the general case, when both backgrounds are nontrivial and $\mathcal{H}_{eff}$ has the form (2.20), we construct the corresponding quantum Hamiltonian in the following way:

$$\hat{H}(\tau) = \hat{\zeta} q \hat{A}_0 + \hat{\Omega} \ , \tag{3.6}$$

where the operator $\hat{A}_0$ is related to the classical quantity $A_0|_{x^0=\zeta\tau}$ and has the following block-diagonal form $\hat{A}_0 = \mathrm{bdiag}\left(A_0|_{x^0=\tau}\ I,\ A_0|_{x^0=-\tau}\ I \right)$, and $\hat{\Omega}$ is an operator related to the classical quantity $\omega|_{x^0=\zeta\tau}$. We define the latter operator as follows

$$\hat{\Omega} = \mathrm{bdiag}\left(\hat{\omega}|_{x^0=\tau}\ ,\ \hat{\omega}|_{x^0=-\tau} \right) \ , \quad \hat{\omega} = \begin{pmatrix} 0 & M \\ G & 0 \end{pmatrix} \ , \tag{3.7}$$

$$M = -\left[\hat{p}_k + qA_k\right] \sqrt{-g} g^{kj} \left[\hat{p}_j + qA_j\right] + m^2 \sqrt{-g},\ G = \frac{g_{00}}{\sqrt{-g}} \ . \tag{3.8}$$

Its square reads $\hat{\Omega}^2 = \mathrm{bdiag}\left(MG|_{x^0=\tau} I, \;\; GM|_{x^0=-\tau} I\right)$, and corresponds (in the classical limit) to the square of the classical quantity $\omega|_{x^0=\zeta\tau}$. A verification of the latter statement may be done, for example, on the states with a definite value of ζ. Natural symmetric operator ordering in the expression for the operator M provides the gauge invariance under $U(1)$ transformations of external electromagnetic field potentials and formal covariance of the theory under general coordinate transformations as will be seen below. One can check that the operator $\hat{H}$ is Hermitian with respect to the inner product (3.3).

The quantum Hamiltonian (3.6) may be written in the following block-diagonal form convenient for the further consideration:

$$\hat{H}(\tau) = \mathrm{bdiag}\left(\hat{h}(\tau), \, -\sigma_3 \hat{h}(-\tau)\sigma_3\right) \, , \;\; \hat{h}(\tau) = \hat{h}(x^0)\Big|_{x^0=\tau} \, , \;\; \hat{h}(x^0) = qA_0 I + \hat{\omega} \, . \tag{3.9}$$

The states of the system under consideration evolute in time τ in accordance with the Schrödinger equation

$$i\hbar\partial_\tau \mathbf{\Psi}(\tau) = \hat{H}(\tau)\mathbf{\Psi}(\tau) \, , \tag{3.10}$$

where the state vectors $\mathbf{\Psi}$ depend now parametrically on τ,

$$\mathbf{\Psi}(\tau) = \begin{pmatrix} \Psi_{+1}(\tau, \mathbf{x}) \\ \Psi_{-1}(\tau, \mathbf{x}) \end{pmatrix}, \;\; \Psi_\zeta(\tau, \mathbf{x}) = \begin{pmatrix} \chi_\zeta(\tau, \mathbf{x}) \\ \varphi_\zeta(\tau, \mathbf{x}) \end{pmatrix}, \;\; \zeta = \pm 1 \, . \tag{3.11}$$

Taking into account the representation (3.9), one can see that two columns $\Psi_z(\tau, \mathbf{x})$, obey the following equations:

$$i\hbar\partial_\tau \Psi_{+1}(\tau, \mathbf{x}) = \hat{h}(\tau)\Psi_{+1}(\tau, \mathbf{x}) \, , \; i\hbar\partial_\tau \Psi_{-1}(\tau, \mathbf{x}) = -\sigma_3 \hat{h}(-\tau)\sigma_3 \Psi_{-1}(\tau, \mathbf{x}) \, . \tag{3.12}$$

Let us now demonstrate that the set of equations (3.12) is equivalent to two Klein-Gordon equations, one for a scalar field of the charge q, and another one for a scalar field of the charge $-q$. In accordance with our classical interpretation we may regard $\hat{\zeta}$ as charge sign operator. Let $\mathbf{\Psi}_\zeta$ be states with a definite charge ζq,

$$\hat{\zeta}\mathbf{\Psi}_\zeta = \zeta \mathbf{\Psi}_\zeta \, , \;\; \zeta = \pm 1 \, . \tag{3.13}$$

It is easily to see that states $\mathbf{\Psi}_{+1}$ with the charge q have $\Psi_{-1} = 0$. In this case $\tau = x^0$, where x^0 is physical time. Then the first equation (3.12) may be rewritten as

$$i\hbar\partial_0 \Psi_{+1}(x^0, \mathbf{x}) = \hat{h}(x^0)\Psi_{+1}(x^0, \mathbf{x}) \,, \quad \Psi_{+1}(x^0, \mathbf{x}) = \begin{pmatrix} \chi_{+1}(x^0, \mathbf{x}) \\ \varphi_{+1}(x^0, \mathbf{x}) \end{pmatrix} . \tag{3.14}$$

Denoting $\varphi_{+1}(x^0, \mathbf{x}) = \varphi(x)$ and remembering the structure (3.7) of the operator $\hat{\omega}$, we get exactly the covariant Klein-Gordon equation in curved space-time for the scalar field $\varphi(x)$ with the charge q,

$$\left[\frac{1}{\sqrt{-g}}\left(i\hbar\partial_\mu - qA_\mu\right)\sqrt{-g}g^{\mu\nu}\left(i\hbar\partial_\nu - qA_\nu\right) - m^2\right]\varphi(x) = 0 \,. \tag{3.15}$$

States $\mathbf{\Psi}_{-1}$ with charge $-q$ have $\Psi_{+1} = 0$. In this case, according to our classical interpretation, $\tau = -x^0$, where x^0 is physical time. Then, taking into account the relation

$$-\left[\sigma_3\hat{h}(x^0)\sigma_3\right]^* = \hat{h}(x^0)\Big|_{q\to -q} = \hat{h}^c(x^0) \,. \tag{3.16}$$

we get from the second equation (3.12)

$$i\hbar\partial_0 \Psi^*_{-1}(-x^0, \mathbf{x}) = \hat{h}^c(x^0)\Psi^*_{-1}(-x^0, \mathbf{x}) \,, \quad \Psi^*_{-1}(-x^0, \mathbf{x}) = \begin{pmatrix} \chi^*_{-1}(-x^0, \mathbf{x}) \\ \varphi^*_{-1}(-x^0, \mathbf{x}) \end{pmatrix} . \tag{3.17}$$

Denoting $\varphi^*_{-1}(-x^0, \mathbf{x}) = \varphi^c(x)$, one may rewrite the equation (3.17) in the form of the covariant Klein-Gordon equation in curved space-time for the charge conjugated scalar field $\varphi^c(x)$ (that which describes particles with the charge $-q$),

$$\left[\frac{1}{\sqrt{-g}}\left(i\hbar\partial_\mu + qA_\mu\right)\sqrt{-g}g^{\mu\nu}\left(i\hbar\partial_\nu + qA_\nu\right) - m^2\right]\varphi^c(x) = 0 \,. \tag{3.18}$$

The inner product (3.3) between two solutions of the Schrödinger equation (3.10) with different charges is zero. For two solutions with charges q it takes the form:

$$\begin{aligned} &\left(\mathbf{\Psi}_{+1}, \mathbf{\Psi}'_{+1}\right) = (\varphi, \varphi')_{KG} \\ &= \int \sqrt{-g}g^{00}\left\{\left[\left(i\hbar\partial_0 - qA_0\right)\varphi\right]^* \varphi' + \varphi^*\left(i\hbar\partial_0 - qA_0\right)\varphi'\right\} d\mathbf{x} \,, \end{aligned} \tag{3.19}$$

it is expressed via Klein-Gordon scalar product on the $x^0 = \text{const}$ hyperplane for the case of the charge q , see (A9). For two solutions with charges $-q$ the inner product (3.3) reads:

$$
\begin{aligned}
&\left(\mathbf{\Psi}_{-1}, \mathbf{\Psi}'_{-1}\right) = (\varphi^c, \varphi^{c'})^c_{KG} \\
&= \int \sqrt{-g} g^{00} \left\{ \left[(i\hbar\partial_0 + qA_0)\,\varphi^c \right]^* \varphi^{c'} + \varphi^{c*} \left(i\hbar\partial_0 + qA_0 \right) \varphi^{c'} \right\} d\mathbf{x}\,,
\end{aligned}
\tag{3.20}
$$

it is expressed via Klein-Gordon scalar product for the case of the charge $-q$, which is denoted above by an upper index c.

Each block-diagonal operator $\hat{T}$, which acts in $\mathcal{R}$, induces operators acting on the fields $\varphi(x)$ and $\varphi^c(x)$. In particular, for the operators $\hat{\mathcal{P}}_k = \hat{\zeta}\hat{P}_k$ of the physical momenta in a flat space-time (see classical interpretation), we get $\hat{\mathcal{P}}_k \mathbf{\Psi} \ \rightarrow \hat{p}_k \ \varphi(x)$ and $\hat{p}_k \ \varphi^c(x)$, as one can expect, since the form of the momentum operators do not depend on the field charge sign.

The above demonstration, together with the previous classical analysis (see Sect.II) has confirmed ones again that x^0 may be treated as physical time. Thus, it is natural to reformulate the evolution in the quantum mechanics constructed in terms of this physical time. At the same time we pass to a different representation of state vectors, taking into account the physical meaning of the components $\Psi_{\pm 1}$, which follows from the equation (3.14) and (3.17). Namely, we will describe quantum mechanical states by means of four columns

$$
\begin{aligned}
&\mathbf{\Psi}(x^0) = \begin{pmatrix} \Psi(x) \\ \Psi_c(x) \end{pmatrix}, \quad \Psi(x) = \Psi_{+1}(x^0, \mathbf{x}), \ \Psi^c(x) = \Psi^*_{-1}(-x^0, \mathbf{x})\,, \\
&\Psi(x) = \begin{pmatrix} \chi(x) \\ \varphi(x) \end{pmatrix}, \quad \Psi^c(x) = \begin{pmatrix} \chi^c(x) \\ \varphi^c(x) \end{pmatrix}.
\end{aligned}
\tag{3.21}
$$

As was said above it is, in fact, a transition to a new representation. Such a representation we may call conditionally x^0-representation, in contrast with the representation (3.2) or (3.11), which will be called τ-representation. The inner product of two states $\mathbf{\Psi}(x^0)$ and $\mathbf{\Psi}'(x^0)$ in x^0 representation takes the form

$$
(\mathbf{\Psi}, \mathbf{\Psi}') = (\Psi, \Psi') + \left(\Psi^c, \Psi^{c\prime}\right)\,,
\tag{3.22}
$$

where the product (Ψ, Ψ') is still given by the equation (3.4).

One may find expressions for the basic operators in x^0-representation under consideration. The operators $\hat{\zeta}$ and $\hat{X}^k$ defined by the expressions (3.5) retain their form, whereas the expressions for the Hamiltonian and the momenta change. The former one has the form

$$\hat{H}(x^0) = \mathrm{bdiag}\left(\hat{h}(x^0)\,, \quad \hat{h}^c(x^0)\right)\,, \tag{3.23}$$

where $\hat{h}(x^0)$ is the corresponding Hamiltonian from (3.9) and $\hat{h}^c(x^0)$ is given by Eq. (3.16). The operator of the physical momentum $\hat{\mathcal{P}}_i = \zeta\hat{P}_i$ has the form $\hat{\mathcal{P}}_i = \hat{p}_k\mathbf{I}$ in x^0 representation. That confirms the interpretation of the physical momentum derived from the classical consideration in the previous Section. The time evolution of state vectors in x^0-representation follows from the equations (3.14) and (3.17)

$$i\hbar\partial_0\mathbf{\Psi}(x^0) = \hat{H}(x^0)\mathbf{\Psi}(x^0)\,. \tag{3.24}$$

IV. FIRST QUANTIZED THEORY AND ONE-PARTICLE SECTOR OF QUANTIZED SCALAR FIELD

Below we will give an interpretation of the quantum mechanics constructed, comparing it with a dynamics of a one-particle sector of QFT of complex scalar field. To this end we are going first to demonstrate that the one-particle sector of the QFT (in cases when it may be consistently defined, see Appendix) may be formulated as a relativistic quantum mechanics without infinite number of negative energy levels and negative norms of state vectors. Then we will show that it may be identified (under certain suppositions) with the quantum mechanics, which was constructed by us in the previous Section in course of the first quantization of the corresponding classical action. Doing that, we may, at the same time, give a more exact interpretation of the quantum mechanics. Below we use widely notions, results, and notations presented in the Appendix.

To begin with one ought to remember that the one-particle sector of QFT (as well as any sector with a definite particle number) may be defined in an unique way for all

time instants only in external backgrounds, which do not create particles from the vacuum [28–32]. Nonsingular time independent external backgrounds give an important example of the above backgrounds, see Appendix. That is why we are going first to present a discussion for such kind of backgrounds to simplify the consideration. A generalization to arbitrary backgrounds, in which the vacuum remains stable, may be done in the similar manner.

Let us reduce the total Fock space $\mathcal{R}^{FT}$ of the QFT to a subspace of vectors which obey the condition $\hat{N}|\mathbf{\Psi}>=|\mathbf{\Psi}>$, where $\hat{N}$ is the operator of number of particles (A37). Namely, we select the subspace $\mathcal{R}^1=\mathcal{R}^{FT}_{10}\oplus\mathcal{R}^{FT}_{01}$ of normalized vectors having the form:

$$|\mathbf{\Psi}>=\sum_n\left[f_n a_n^+ + \lambda_n b_n^+\right]|0>\,, \tag{4.1}$$

where $f_n,\ \lambda_n$ are arbitrary coefficients, $\sum_n\left[|f_n|^2+|\lambda_n|^2\right]<\infty$. We are going to call $\mathcal{R}^1$ one-particle sector of QFT. All state vectors from the one-particle sector (as well as any vector from the Fock space) have positive norms.

The vectors $|n,\zeta>$ form a complete basis in $\mathcal{R}^1$,

$$|n,\zeta>=\left\{\begin{array}{l} a_n^+|0>,\ \zeta=+1\,, \\ b_n^+|0>,\ \zeta=-1. \end{array}\right. \tag{4.2}$$

The spectrum of the Hamiltonian $\hat{H}^{FT}_R$ (see (A35)) in the space $\mathcal{R}^1$ reproduces exactly one-particle energy spectrum of particles and antiparticles without infinite number of negative energy levels,

$$\hat{H}^{FT}_R|n,+1>=\epsilon_{+,n}|n,+1>,\ \hat{H}^{FT}_R|n,-1>=\epsilon^c_{+,n}|n,-1>\ . \tag{4.3}$$

The dynamics of the one-particle sector may be formulate as a relativistic quantum mechanics under certain suppositions. To demonstrate that, we pass first to a coordinate representation for state vectors of the QFT, which is an analog of common in nonrelativistic quantum mechanics coordinate representation. Consider the decompositions

$$\hat{\psi}(x)=\hat{\psi}_{(-)}(x)+\hat{\psi}_{(+)}(x),\ \ \hat{\psi}^c(x)=-\left(\hat{\psi}^+\sigma_3\right)^T=\hat{\psi}^c_{(-)}(x)+\hat{\psi}^c_{(+)}(x)\,,$$

$$
\begin{aligned}
&\hat{\psi}_{(-)}(x)=\sum_n a_n\psi_{+,n}(x),\ \hat{\psi}_{(+)}(x)=\sum_\alpha b_\alpha^+\psi_{-,\alpha}(x)\ ,\\
&\hat{\psi}^c_{(-)}(x)=-\sum_\alpha b_\alpha\sigma_3\psi^*_{-,\alpha}(x)=\sum_\alpha b_\alpha\psi^c_{+,\alpha}(x)\ ,\\
&\hat{\psi}^c_{(+)}(x)=-\sum_n a_n^+\sigma_3\psi^*_{+,n}(x)=\sum_n a_n^+\psi^c_{-,n}(x)\ ,
\end{aligned} \tag{4.4}
$$

where $\hat{\psi}^c(x)$ is charge conjugated Heisenberg operator of the field $\hat{\psi}(x)$. By means of such defined operators we may construct a ket basis in the Fock space $\mathcal{R}^{FT}$,

$$
<0|\hat{\psi}_{(-)}(\mathbf{x}_1)\ldots\hat{\psi}_{(-)}(\mathbf{x}_A)\hat{\psi}^c_{(-)}(\mathbf{y}_1)\ldots\hat{\psi}^c_{(-)}(\mathbf{y}_B)\,,\ \ A,B=0,1,\ldots\ , \tag{4.5}
$$

where

$$
\hat{\psi}_{(-)}(\mathbf{x})=\left.\hat{\psi}_{(-)}(x)\right|_{x^0=0},\ \ \hat{\psi}^c_{(-)}(\mathbf{y})=\left.\hat{\psi}^c_{(-)}(y)\right|_{y^0=0}\ .
$$

Then a time-dependent state vector $|\mathbf{\Psi}(x^0)>$ from the Fock space (in the coordinate representation) is given by a set of its components

$$
\begin{aligned}
&\Psi_{AB}(x_1\ldots x_A,y_1\ldots y_B)=<0|\hat{\psi}_{(-)}(\mathbf{x}_1)\ldots\hat{\psi}_{(-)}(\mathbf{x}_A)\hat{\psi}^c_{(-)}(\mathbf{y}_1)\ldots\hat{\psi}^c_{(-)}(\mathbf{y}_B)|\mathbf{\Psi}(x^0)>\\
&=<0|\hat{\psi}_{(-)}(x_1)\ldots\hat{\psi}_{(-)}(x_A)\hat{\psi}^c_{(-)}(y_1)\ldots\hat{\psi}^c_{(-)}(y_B)|\mathbf{\Psi}(0)>\ ,\\
&x_1^0=\ldots=x_A^0=y_1^0=\ldots=y_B^0=x^0\ .
\end{aligned} \tag{4.6}
$$

Let us consider a time-dependent state $|\mathbf{\Psi}(x^0)>$ from the subspace $\mathcal{R}^1$. It has only nonzero components $\Psi_{10}(x)$ and $\Psi_{01}(x)$,

$$
\begin{aligned}
&\Psi_{10}(x)=<0|\hat{\psi}_{(-)}(x)|\mathbf{\Psi}(0)>=<0|\hat{\psi}(x)|\mathbf{\Psi}(0)>\ ,\\
&\Psi_{01}(x)=<0|\hat{\psi}^c_{(-)}(x)|\mathbf{\Psi}(0)>=<0|\hat{\psi}^c(x)|\mathbf{\Psi}(0)>\ \ .
\end{aligned} \tag{4.7}
$$

Thus, one may describe states from $\mathcal{R}^1$ in the coordinate representation by four columns

$$
\begin{aligned}
&\mathbf{\Psi}(x^0)=\begin{pmatrix}\Psi(x)\\ \Psi^c(x)\end{pmatrix},\ \Psi(x)=\Psi_{10}(x),\ \Psi^c(x)=\Psi_{01}(x),\\
&\Psi(x)=\begin{pmatrix}\chi(x)\\ \varphi(x)\end{pmatrix},\ \Psi^c(x)=\begin{pmatrix}\chi^c(x)\\ \varphi^c(x)\end{pmatrix}\ .
\end{aligned} \tag{4.8}
$$

The QFT inner product $<\boldsymbol{\Psi}|\boldsymbol{\Psi}'>$ of two states from $\mathcal{R}^1$ may be written via their representatives in the coordinate representation. To this end one may use the following expression for the projection operator to one-particle sector

$$\int\left[\hat{\overline{\psi}}(\mathbf{x})|0><0|\hat{\psi}(\mathbf{x})+\hat{\overline{\psi^c}}(\mathbf{x})|0><0|\hat{\psi}^c(\mathbf{x})\right]d\mathbf{x}=I\ . \tag{4.9}$$

It follows from the relation (A17) and properties of the solutions $\psi_{\zeta,n}(x)$. Then the inner product $(\boldsymbol{\Psi},\boldsymbol{\Psi}')$ of two states $\boldsymbol{\Psi}(x^0)$ and $\boldsymbol{\Psi}'(x^0)$ from the one-particle sector in the coordinate representation may be written as

$$\begin{aligned}&(\boldsymbol{\Psi},\boldsymbol{\Psi}')=(\Psi,\Psi')+\left(\Psi^c,\Psi^{c\prime}\right),\ \left((\boldsymbol{\Psi},\boldsymbol{\Psi}')=<\boldsymbol{\Psi}|\boldsymbol{\Psi}'>\right),\\&(\Psi,\Psi')=\int\overline{\Psi}(x)\Psi(x)d\mathbf{x}=\int\left[\chi^*(x)\varphi'(x)+\varphi^*(x)\chi'(x)\right]d\mathbf{x}\ .\end{aligned} \tag{4.10}$$

One may find expressions for the basic operators in the coordinate representation in the one-particle sector:

$$\hat{H}_R^{FT}\to\hat{H}=\mathrm{bdiag}\left(\hat{h}\,,\ \hat{h}^c\right)\ , \tag{4.11}$$

where $\hat{h}$ and $\hat{h}^c$ are defined by Eq. (3.9) and (3.16), so that (4.11) is, in fact, the quantum mechanical Hamiltonian (3.23) in the case under consideration (in time-independent backgrounds); $\hat{P}_k^{FT}\to\hat{P}_k=\mathrm{bdiag}\left(P_k,\ P_k^c\right)$, where the operator P_k is defined by the expression (A1), $P_k=i\hbar\partial_k-qA_k$, and $P_k^c=P_k|_{q\to-q}=-P_k^*=i\hbar\partial_k+qA_k$; $\hat{Q}^{FT}\to\hat{Q}=q\hat{\zeta}$, where $\hat{\zeta}$ is the operator from (3.5).

An analog of the equations (4.3) in the coordinate representation has the form

$$\begin{aligned}&\hat{H}\boldsymbol{\Psi}_{+,n}=\epsilon_{+,n}\boldsymbol{\Psi}_{+,n},\quad\boldsymbol{\Psi}_{+,n}=\begin{pmatrix}\psi_{+,n}(\mathbf{x})\\0\end{pmatrix},\\&\hat{H}\boldsymbol{\Psi}_{+,\alpha}^c=\epsilon_{+,\alpha}^c\boldsymbol{\Psi}_{+,\alpha}^c,\quad\boldsymbol{\Psi}_{+,\alpha}^c=\begin{pmatrix}0\\\psi_{+,\alpha}^c(\mathbf{x})\end{pmatrix},\\&\left(\boldsymbol{\Psi}_{+,n},\boldsymbol{\Psi}_{+,m}\right)=\delta_{nm},\left(\boldsymbol{\Psi}_{+,\alpha}^c,\boldsymbol{\Psi}_{+,\beta}^c\right)=\delta_{\alpha\beta},\left(\boldsymbol{\Psi}_{+,n},\boldsymbol{\Psi}_{+,\alpha}^c\right)=0,\end{aligned} \tag{4.12}$$

see (A25), (A27), and (A29). The set $\boldsymbol{\Psi}_{+,n},\ \boldsymbol{\Psi}_{+,\alpha}^c$ forms a complete basis in $\mathcal{R}^1$ in the coordinate representation.

The time evolution of state vectors from the one-particle sector in the coordinate representation may be found using the equations (A33). Thus, one may write

$$i\hbar\partial_0\Psi(x) = \hat{h}\Psi(x)\,, \;\; i\hbar\partial_0\Psi^c(x) = \hat{h}^c\Psi^c(x)\;, \tag{4.13}$$

or using the notations introduced above

$$i\hbar\partial_0\boldsymbol{\Psi}(x^0) = \hat{H}\boldsymbol{\Psi}(x^0)\;. \tag{4.14}$$

According to superselection rules [6] physical states are only those which are eigenvectors for the charge operator (A35). Thus, among the vectors (4.1) only those, which obey the condition

$$\hat{Q}^{FT}|\boldsymbol{\Psi}>=\zeta q|\boldsymbol{\Psi}>, \;\; \zeta = \pm 1\;, \tag{4.15}$$

are physical. This condition defines a physical subspace $\mathcal{R}^1_{ph}$ from the one-particle sector. It is easy to see that $\mathcal{R}^1_{ph} = \mathcal{R}^{FT}_{10} \cup \mathcal{R}^{FT}_{01}$. Vectors from the physical subspace of the one-particle sector have the form:

$$|\boldsymbol{\Psi}>=\left(\sum_n f_n a_n^+|0>, \;\sum_n \lambda_n b_n^+|0>\right)\;, \tag{4.16}$$

where $f_n,\ \lambda_n$ are arbitrary coefficients, $\sum_n |f_n|^2 < \infty,\ |\lambda_n|^2 < \infty$. Since the charge operator has the block-diagonal form in the one-particle sector in the coordinate representation (see above), the condition (4.15) reads

$$\hat{\zeta}\boldsymbol{\Psi}_\zeta = \zeta\boldsymbol{\Psi}_\zeta, \;\; \zeta = \pm 1\;. \tag{4.17}$$

Thus, the physical subspace of the one-particle sector consists of the vectors $\boldsymbol{\Psi}_\zeta,\ \ \zeta = \pm 1$ only. Due to the structure of the operator $\hat{\zeta}$, the states $\boldsymbol{\Psi}_{+1}$ contain only the upper half of components, whereas, ones $\boldsymbol{\Psi}_{-1}$ contain only the lower half of components,

$$\boldsymbol{\Psi}_{+1} = \begin{pmatrix} \Psi(\mathbf{x}) \\ 0 \end{pmatrix}, \;\; \boldsymbol{\Psi}_{-1} = \begin{pmatrix} 0 \\ \Psi^c(\mathbf{x}) \end{pmatrix}\;. \tag{4.18}$$

One may see that the complete set $\mathbf{\Psi}_{+,n}$ and $\mathbf{\Psi}^c_{+,\alpha}$ from (4.12) consists only of physical vectors.

The continuity equation, which follows from (4.14), has the form

$$
\begin{aligned}
&\partial_0\rho + \mathrm{div}\mathbf{j} = 0,\ \rho = \overline{\Psi}\Psi + \overline{\Psi^c}\Psi^c\ , \\
&j^i = \frac{1}{2}g^{ik}\sqrt{-g}\left\{\left[\overline{\Psi}P_k + \overline{(P_k\Psi)}\right](\sigma_1 + i\sigma_2)\Psi + \left[\overline{\Psi^c}P^c_k + \overline{(P^c_k\Psi^c)}\right](\sigma_1 + i\sigma_2)\Psi^c\right\}\ . \qquad (4.19)
\end{aligned}
$$

Let us denote via $\rho_\zeta(x)$ the quantity (4.19) constructed from the physical states $\mathbf{\Psi}_\zeta$. This quantity may not be interpreted as a probability density, since it is not positively defined in general case. That is a reflection of a well-known fact that one cannot construct one-particle localized states in relativistic quantum theory. However, due to the Eq. (4.10) we get $\int \rho_\zeta d\mathbf{x} = (\mathbf{\Psi}_\zeta, \mathbf{\Psi}_\zeta) = 1$ for any normalized physical states. Moreover, the overlap $\left(\mathbf{\Psi}_\zeta, \mathbf{\Psi}'_\zeta\right)$ may be treated as a probability amplitude, supporting usual quantum mechanical interpretation.

Summarizing all what was said above, we may conclude that, in fact, the QFT dynamics in the physical subspace of the one-particle sector in the coordinate representation is formulated as a consistent relativistic quantum mechanics. It does not meet well-known difficulties usual for standard formulations of the relativistic quantum mechanics of spinless particles [2,3,5] such as negative norms and infinite number of negative energy levels.

Now we may return to the interpretation of the results of the first quantization presented in the previous Section. In the time independent nonsingular backgrounds under consideration, we may see that under certain restrictions our quantum mechanics coincides literally with the dynamics of the QFT in the physical subspace of the one-particle sector. These restrictions are related only to an appropriate definition of the Hilbert space of the quantum mechanics. Indeed, all other constructions in the quantum mechanics and in the one-particle sector of the QFT in the coordinate representation coincide. The space $\mathcal{R}$, in which the commutation relations (3.1) were realized (the space of the vectors of the form (3.2)), is too wide, in particular, it contains negative norm vectors. We have first to restrict it to a subspace, which is equivalent to $\mathcal{R}^1$ and after that to the physical subspace $\mathcal{R}^1_{ph}$.

Thus, we may get a complete equivalence between the both theories. To do the first step we consider the eigenvalue problem for the Hamiltonian (3.23) in the space $\mathcal{R}$. Its spectrum is wider than one (4.12) in the space $\mathcal{R}^1$,

$$
\begin{gathered}
\hat{H}\boldsymbol{\Psi}_{\varkappa,n}=\epsilon_{\varkappa,n}\boldsymbol{\Psi}_{\varkappa,n},\;\boldsymbol{\Psi}_{\varkappa,n}=\left(\begin{array}{c}\psi_{\varkappa,n}(\mathbf{x})\\ 0\end{array}\right),\;(\boldsymbol{\Psi}_{\varkappa,n},\boldsymbol{\Psi}_{\varkappa',m})=\varkappa\delta_{\varkappa\varkappa'}\delta_{nm}\,,\\
\hat{H}\boldsymbol{\Psi}^c_{\varkappa,\alpha}=\epsilon^c_{\varkappa,\alpha}\boldsymbol{\Psi}^c_{\varkappa,\alpha},\;\boldsymbol{\Psi}^c_{\varkappa,\alpha}=\left(\begin{array}{c}0\\ \psi^c_{\varkappa,\alpha}(\mathbf{x})\end{array}\right),\;\left(\boldsymbol{\Psi}^c_{\varkappa,\alpha},\boldsymbol{\Psi}^c_{\varkappa',\beta}\right)=\varkappa\delta_{\varkappa\varkappa'}\delta_{\alpha\beta}\,,\\
\left(\boldsymbol{\Psi}_{\varkappa,n},\boldsymbol{\Psi}^c_{\varkappa',\alpha}\right)=0\,,\varkappa,\varkappa'=\pm\,,
\end{gathered}
\tag{4.20}
$$

see (A25), (A27), and (A29). To get the same spectrum as in QFT, we need to eliminate all the vectors $\boldsymbol{\Psi}_{-,n}$ and $\boldsymbol{\Psi}^c_{-,\alpha}$ from the consideration. Thus, we may define the analog of the space $\mathcal{R}^1$ as a linear envelop of the vectors $\boldsymbol{\Psi}_{+,n}$ and $\boldsymbol{\Psi}^c_{+,\alpha}$ only. This space does not contain negative norm vectors and the operator $\hat{\Omega}$ from (3.7) is positively defined in perfect accordance with the positivity of the classical quantity ω. The spectrum of the Hamiltonian in such defined space coincides with one of the Hamiltonian of the QFT in the one-particle sector. Reducing $\mathcal{R}^1$ to $\mathcal{R}^1_{ph}$, we get literal coincidence between both theories.

One ought to mention a well-know in the relativistic quantum mechanics problem of position operator construction (see [1–4] and references there). In all the works where they started with a given K-G or Dirac equation as a Schrödinger ones, the construction of such an operator was a heuristic task. The form of the operator had to be guessed to obey some physical demands, by analogy with the nonrelativistic case. In particular, an invariance of the one-particle sector with a given sign of energy under the action of the position operator was expected. Besides, mean values of the operator had to have necessary transformation properties under the coordinate transformations and the correspondence principle had to hold. Realizing these and some other demands they met serious difficulties. At present, from the position of a more deep understanding of the quantum field theory, it is clear that it is impossible to construct localized one-particle states. That means that the position operator with the above mentioned properties does not exist. In the frame of our consideration, which

starts with a given classical theory, the coordinate x becomes an operator $\hat{X}$ in course of the quantization . Thus, the correspondence principle holds automatically. We do not demand from the operator $\hat{X}$ literally similar properties as in non-relativistic quantum mechanics. In particular, the one-particle sector is not invariant under the action of such an operator. The operator has no eigenvectors in this sector. To construct such eigenvectors one has to include into the consideration many-particle states. In the present work we do not exceed the limits of the one-particle consideration. However, a generalization to many-particle theory may be done one the base of the constructed one-particle sector and the existence of eigenvectors of the operator $\hat{X}$ may be demonstrated. That will be presented in our next article. As to the momentum operator, similar problem appears only in nonuniform external backgrounds, and has to be understood similarly.

The above comparison of the first quantized theory with the dynamics of the one-particle sector of QFT was done for non-singular and time-independent external backgrounds. It may be easily extended to any time-dependent background, which do not create particles from vacuum.

Thus, we see that the first quantization of a classical action leads to a relativistic quantum mechanics which is consistent to the same extent as quantum field theory in the one-particle sector. Such a quantum mechanics describes spinless charged particles of both signs, and reproduces correctly their energy spectra, which is placed on the upper half-plane of the Fig.1 (see App.).

One may think that the reduction of the space $\mathcal{R}$ of the quantum mechanics to the space $\mathcal{R}^1$ is necessary only in the first quantization, thus an equivalence between the first and the second quantization is not complete. That may be interpreted as a weak point in the presented scheme of the first quantization. However, it is a wrong impression, the same procedure is present in the second quantization. Below we are going to remember how it happens in the case under consideration of scalar field. Indeed, instead to write the decomposition (A34) one could write two possible decompositions

$$\hat{\psi}(x)=\sum_{\varkappa n}a_{\varkappa,n}\psi_{\varkappa,n}(x)\,,\quad \hat{\psi}^c(x)=\sum_{\varkappa n}b_{\varkappa,n}\psi^c_{\varkappa,n}(x)\,,$$

since both sets $\psi_{\varkappa,n}$ and $\psi^c_{\varkappa,n}$ are complete. Then it follows from the commutation relations (A32) that

$$[a_{\varkappa,n},a^+_{\varkappa',m}]=[b_{\varkappa,n},b^+_{\varkappa',m}]=\varkappa\delta_{\varkappa\varkappa'}\delta_{nm},\ [a_{\varkappa,n},a_{\varkappa',m}]=[b_{\varkappa,n},b_{\varkappa',m}]=0\,.$$

Two sets of operators a,a^+ and b,b^+ are not independent, they are related as follows: $a_{+,n}=b^+_{-,n}\,,\ a^+_{+,n}=b_{-,n}\,,\ a_{-,n}=b^+_{+,n}\,,\ a^+_{-,n}=b_{+,n}$. Interpreting the operators without cross as annihilation ones, we may define four vacuum vectors

$$|0\rangle=|a_+\rangle\otimes|b_+\rangle\,,\ |0\rangle_1=|a_+\rangle\otimes|a_-\rangle\,,\ |0\rangle_2=|b_+\rangle\otimes|b_-\rangle\,,\ |0\rangle_3=|a_-\rangle\otimes|b_-\rangle\,,$$

where $a_{+,n}|a_+>=0,\ a_{-,n}|a_->=0,\ b_{+,n}|b_+>=0,\ b_{-,n}|b_->=0$, and the following one-particle excited states:

$$1)\ a^+_{+,n}|0\rangle\,,\ b^+_{+,n}|0\rangle\,;\ 2)\ \ a^+_{+,n}|0\rangle_1\,,\ a^+_{-,n}|0\rangle_1\,;\ 3)\ b^+_{+,n}|0\rangle_2\,,\ b^+_{-,n}|0\rangle_2\,;\ 4)\ \ a^+_{-,n}|0\rangle_3\,,\ b^+_{-,n}|0\rangle_3\,.$$

The non-renormalized quantum Hamiltonian, which may be constructed from the classical expression (A14), reads $\hat{H}^{FT}=\sum_{\varkappa,n}\varkappa\epsilon_{\varkappa,n}a^+_{\varkappa,n}a_{\varkappa,n}$. Now one may see that only the one-particle states from the group 1 form the physical subspace. All other states from the groups 2,3,4 have to be eliminated, since they or contain negative energy levels, negative norms, or do not reproduce complete spectrum of particles and antiparticles. Working in such defined physical subspace we may deal only with the operators $a^+_{+,n},a_{+,n},b^+_{+,n},b_{+,n}$ and denote them simply as a^+_n,a_n,b^+_n,b_n. Then all usual results of second quantized theory may be reproduced.

Finally, to complete the consideration let us examine the Dirac quantization of the theory in question. In this case we do not need to impose any gauge condition to the first-class constraint (2.7). We assume as before the operator $\hat{\zeta}$ to have the eigenvalues $\zeta=\pm1$ by analogy with the classical theory. The equal time commutation relations for the operators $\hat{X}^\mu,\hat{P}_\mu,\hat{\zeta}$, which correspond to the variables x^μ,p_μ,ζ, we define according to their Poisson brackets, due to the absence of second-class constraints. Thus, now we get for nonzero commutators

$$[\hat{X}^\mu, \hat{P}_\nu] = i\hbar\delta^\mu_\nu \ , \quad \left(\hat{\zeta}^2 = 1\right) . \tag{4.21}$$

Besides, we have to keep in mind the necessity to construct an operator realization for the first class-constraint (2.7), which contains a square root. Taking all that into account, we select as a state space one whose elements $\mathbf{\Psi}$ are x-dependent four-component columns

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_{+1}(x) \\ \Psi_{-1}(x) \end{pmatrix} \ , \quad \Psi_\zeta(x) = \begin{pmatrix} \chi_\zeta(x) \\ \varphi_\zeta(x) \end{pmatrix} \ , \tag{4.22}$$

where $\Psi_\zeta(x), \ \zeta = \pm 1$ are two component columns, with χ and φ being x-dependent functions. We seek all the operators in the block-diagonal form,

$$\hat{\zeta} = \mathrm{bdiag}\,(I, \ -I) \ , \ \ \hat{X}^\mu = x^\mu \mathbf{I} \ , \quad \hat{P}_\mu = \hat{p}_\mu \mathbf{I} \ , \ \ \hat{p}_\mu = -i\hbar\partial_\mu \ . \tag{4.23}$$

where I and $\mathbf{I}$ are 2×2 and 4×4 unit matrices respectively. The operator $\hat{\Phi}_1$ which corresponds to the first-class constraint (2.7), is selected as $\hat{\Phi}_1 = \hat{P}_0 + q\hat{A}_0 + \hat{\zeta}\hat{\Omega}$. The operators $\hat{A}_0$ and $\hat{\Omega}$ related to the classical quantities A_0 and ω have the following forms $\hat{A}_0 = A_0\mathbf{I}$, $\hat{\Omega} = \mathrm{bdiag}\,(\hat{\omega}, \hat{\omega})$, where $\hat{\omega}$ is defined by Eq. (3.7). Similar to the canonical quantization case, one may verify that the square $\hat{\Omega}^2$ corresponds (in the classical limit) to the square of the classical quantity ω. The state vectors (4.22) do not depend on "time" τ since the Hamiltonian vanishes on the constraints surface. The physical state vectors have to obey the equation $\hat{\Phi}_1\mathbf{\Psi} = 0$. Thus, we arrive to the equations

$$i\hbar\partial_0\Psi_{+1}(x) = (qA_0 + \hat{\omega})\,\Psi_{+1}(x) \ , \ \ i\hbar\partial_0\Psi_{-1}(x) = (qA_0 - \hat{\omega})\,\Psi_{-1}(x) \ . \tag{4.24}$$

Taking into account the realization of all the operators, definitions (3.9), (3.16), and denoting $\Psi_{+1}(x) = \psi(x)$, $-\sigma_3\Psi^*_{-1}(x) = \psi^c(x)$, we get two Klein-Gordon equations (in the Hamiltonian form)

$$i\partial_0\psi = \hat{h}(x^0)\psi \ , \ \ i\partial_0\psi^c = \hat{h}^c(x^0)\psi^c \, , \tag{4.25}$$

one for particle, and one for the antiparticle, see (A14) and (A19). Unfortunately, the Dirac method of the quantization gives no more information how to proceed further with

the consistent quantum theory construction, and moreover contains principal contradictions, see discussion in the Introduction. However, we may conclude that at least one of the main feature of the quantum theory, its charge conjugation invariance, remains also in the frame of the Dirac quantization.

V. SPINNING PARTICLE CASE

We would like to demonstrate here that a consistent quantization, similar to that for spinless particle, applied to an action of spinning particle, allows one to construct a consistent relativistic quantum mechanics, which is equivalent to one-particle sector of quantized spinor field. For simplicity we restrict ourselves here only by one external electromagnetic background, considering the problem in the flat space-time.

An action of spin one half relativistic particle (spinning particle), with spinning degrees of freedom describing by anticommuting (Grassmann-odd) variables, was first proposed by Berezin and Marinov [33] and just after that discussed and studied in detail in papers [34–38]. It may be written in the following form (in the flat space-time)

$$S=\int_0^1 Ld\tau\,,\quad L=-\frac{\left(\dot{x}^\mu-i\xi^\mu\chi\right)^2}{2e}-q\dot{x}^\mu A_\mu+iqeF_{\mu\nu}\xi^\mu\xi^\nu-im\xi^4\chi-\frac{e}{2}m^2-i\xi_n\dot{\xi}^n\,, \quad (5.1)$$

where x^μ, e are even and ξ^n, χ are odd variables, dependent on a parameter $\tau\in[0,1]$, which plays a role of time in this theory, $\mu=\overline{0,3}$; $n=(\mu,4)=\overline{0,4}$; $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$; $\eta_{mn}=\mathrm{diag}(1,-1,-1,-1,-1)$. Spinning degrees of freedom are described by odd variables ξ^n; even e and odd χ play an auxiliary role to make the action reparametrization and super gauge-invariant as well as to make it possible consider both cases $m\neq0$ and $m=0$ on the same foot.

The are two types of gauge transformations under which the action (5.1) is invariant: reparametrizations $\delta x^\mu=\dot{x}^\mu\varepsilon$, $\delta e=\frac{d}{d\tau}(e\varepsilon)$, $\delta\xi^n=\dot{\xi}^n\varepsilon$, $\delta\chi=\frac{d}{d\tau}(\chi\varepsilon)$, and supertransformations $\delta x^\mu=i\xi^\mu\epsilon$, $\delta e=i\chi\epsilon$, $\delta\chi=\dot{\epsilon}$, $\delta\xi^\mu=\frac{1}{2e}\left(\dot{x}^\mu-i\xi^\mu\chi\right)\epsilon$, $\delta\xi^4=\frac{m}{2}\epsilon$, where $\varepsilon(\tau)$ and $\epsilon(\tau)$ are τ-dependent gauge parameter, the first one is even and the second one is odd.

Going over to Hamiltonian formulation, we introduce the canonical momenta:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{\dot{x}_\mu - i\xi_\mu \chi}{e} - qA_\mu\,,\ P_e = \frac{\partial L}{\partial \dot{e}} = 0\,,\ P_\chi = \frac{\partial_r L}{\partial \dot{\chi}} = 0\,,\ \pi_n = \frac{\partial_r L}{\partial \dot{\xi}^n} = -i\xi_n\,. \quad (5.2)$$

It follows from (5.2) that there exist primary constraints $\phi^{(1)} = 0$,

$$\phi_1^{(1)} = P_\chi\,,\ \phi_2^{(1)} = P_e\,,\ \phi_{3,n}^{(1)} = \pi_n + i\xi_n\,. \quad (5.3)$$

We construct the total Hamiltonian $H^{(1)} = H + \lambda_a \phi_a^{(1)}$, according to standard procedure [20–22],

$$H = -\frac{e}{2}\left[(p+qA)^2 - m^2 + 2iqF_{\mu\nu}\xi^\mu\xi^\nu\right] + i\left[(p_\mu + qA_\mu)\,\xi^\mu + m\xi^4\right]\chi\,. \quad (5.4)$$

From the conditions of the conservation of the primary constraints $\phi^{(1)}$ in the time τ, $\dot{\phi}^{(1)} = \left\{\phi^{(1)}, H^{(1)}\right\} = 0$, we find secondary constraints $\phi^{(2)} = 0$,

$$\phi_1^{(2)} = (p_\mu + qA_\mu)\xi^\mu + m\xi^4\,,\ \ \phi_2^{(2)} = (p+qA)^2 - m^2 + 2iqF_{\mu\nu}\xi^\mu\xi^\nu\,, \quad (5.5)$$

and determine λ, which correspond to the primary constraint $\phi_{3n}^{(1)}$. Thus, the Hamiltonian H appears to be proportional to the constraints, as one could expect in the case of a reparametrization invariant theory, $H = -\frac{e}{2}\phi_2^{(2)} + i\phi_1^{(2)}\chi$. No more secondary constraints arise from the Dirac procedure, and the Lagrange multipliers, corresponding to the primary constraints $\phi_1^{(1)}$, $\phi_2^{(1)}$, remain undetermined.

One can go over from the initial set of constraints $\phi^{(1)}, \phi^{(2)}$ to the equivalent one $\phi^{(1)}, T$, where

$$T_1 = (p_\mu + qA_\mu)\,(\pi^\mu - i\xi^\mu) + m\left(\pi^4 - i\xi^4\right)\,, \quad (5.6)$$

$$T_2 = p_0 + qA_0 + \zeta r\,,\ \ r = \sqrt{m^2 + (p_k + qA_k)^2 + 2qF_{\mu\nu}\xi^\mu\pi^\nu}\,. \quad (5.7)$$

The new set of constraints can be explicitly divided in a set of the first-class constraints, which is $\phi_{1,2}^{(1)}$, T, and in a set of second-class constraints, which is $\phi_{3,n}^{(1)}$,

$$\left\{\phi_a^{(1)}, \phi^{(1)}\right\} = \left\{\phi_a^{(1)}, T\right\} = \left\{T, \phi_{3,n}^{(1)}\right\}\Big|_{\phi=T=0} = \{T, T\}|_{\phi=T=0} = 0\,,\ \ a = 1, 2\,. \quad (5.8)$$

The constraint (5.7) is equivalent to one $\phi_2^{(2)}$, $\phi_2^{(2)} = -2\zeta r T_2 + (T_2)^2$. Remember, that $\zeta = -\mathrm{sign}\,[p_0 + qA_0(x)]$. Thus, the constraint (5.7) is a analog of the linearized primary constraint (2.7) in the scalar particle case.

We are going to impose supplementary gauge conditions to all the first-class constraints. First we impose two gauge conditions $\phi^G = 0$,

$$\phi_1^G = \pi^0 - i\xi^0 + \zeta\left(\pi^4 - i\xi^4\right), \quad \phi_2^G = x^0 - \zeta\tau \ . \tag{5.9}$$

A motivation for the gauge condition ϕ_2^G is the same as in scalar particle case (Sect. II) . As to the gauge condition ϕ_1^G, it is chosen to be a contrpart to one of the first-class constraint T , and to provide a simple structure of the final complete set of second-class constraints, see below. It differs from similar gauge condition, which was used in [12], by a combination of constraints. From the consistency condition $\dot{\phi}^G = 0$ we find two additional constraints

$$\phi_3^G = \chi - \frac{i\zeta qF_{k0}\left(\pi^k - i\xi^k\right)}{\tilde{\omega}_0(\tilde{\omega}_0 + m)} = 0\ , \tag{5.10}$$

$$\phi_4^G = e - \frac{1}{\tilde{\omega}}\left[1 - \frac{iqF_{k0}\left(\pi^k - i\xi^k\right)\left(\pi^0 - i\xi^0\right)}{2\tilde{\omega}_0(\tilde{\omega}_0 + m)}\right] = 0, \tag{5.11}$$

where

$$\tilde{\omega}_0 = \sqrt{m^2 + (p_k + qA_k)^2 + 2qF_{kl}\xi^k\pi^l}\ ,\ \tilde{\omega} = \sqrt{\tilde{\omega}_0^2 + \frac{2\zeta qF_{k0}}{\tilde{\omega}_0 + m}\left(p_l + qA_l\right)\left(\xi^k\pi^l + \pi^k\xi^l\right)}\ . \tag{5.12}$$

Then, the conditions of consistency for the constraints of ϕ_3^G and ϕ_4^G lead to the determination of the Lagrange multipliers for the primary constraints $\phi_1^{(1)}$ and $\phi_2^{(1)}$. The complete set of constraints $\left(\phi^{(1)}, T, \phi^G\right)$ is already a second-class one.

Below we are going to present an equivalent to $\left(\phi^{(1)}, T, \phi^G\right)$ set of second-class constraints Φ_a, $a = 1, 2, ..., 13$, which has a simple quasi-diagonal matrix $\{\Phi_a, \Phi_b\}$. The first five constraints of this set have the form

$$\Phi_1 = p_0 + qA_0 + \zeta\tilde{\omega}\ ,\ \Phi_2 = \phi_2^G\ ,\ \Phi_3 = \phi_{3,1}^{(1)}\ ,\ \Phi_4 = \phi_{3,2}^{(1)}\ ,\quad \Phi_5 = \phi_{3,3}^{(1)}\ . \tag{5.13}$$

Four of them are exactly old constraints, and the first one is a linear combination of the old constraints. Namely, $\Phi_1 = t_1 T_1 + t_2 T_2 + f\phi_1^G + f_{nm}\phi_{3,m}^{(1)}\phi_{3,n}^{(1)}$, where the coefficient functions are

$$t_1 = \frac{-i\zeta qF_{k0}\left(\pi^k - i\xi^k\right)}{(m+\tilde{\omega}_0)(p_0+qA_0-\zeta\tilde{\omega}\,)}\ ,\quad f = \frac{imqF_{k0}(\pi^k - i\xi^k)}{(m+\tilde{\omega}_0)(p_0+qA_0-\zeta\tilde{\omega}\,)}\ ,$$
$$t_2 = \frac{1}{p_0+qA_0-\zeta\tilde{\omega}}\left[p_0+qA_0-\zeta r\ + \frac{i\zeta qF_{k0}\left(\pi^k - i\xi^k\right)\left(\pi^0 - i\xi^0\right)}{(m+\tilde{\omega}_0)}\right]\ ,$$
$$f_{k,0} = \frac{iqF_{k0}}{p_0+qA_0-\zeta\tilde{\omega}}\left[1 - \frac{iqF_{l0}\left(\pi^l - i\xi^l\right)\left(\pi^0 - i\xi^0\right)}{2(m+\tilde{\omega}_0)\tilde{\omega}_0}\right]\ ,\ f_{k,l} = \frac{i\zeta qF_{k0}\left(p_l+qA_l\right)}{(m+\tilde{\omega}_0)(p_0+qA_0-\zeta\tilde{\omega}\,)}\ .$$

The rest constraints are orthogonal (in sense of the Poisson brackets) to the latter five and form four orthogonal to each other pairs. The first pair is $\Phi_6,\ \Phi_7$, where $\Phi_6 = -\frac{i}{2}T_1 + bT_2 + c\phi_2^G + r_k\phi_{3,k}^{(1)}\phi_{3,0}^{(1)},\ \ \Phi_7 = \phi_1^G$, and

$$b = \frac{i\left\{\phi_2^G, T_1\right\}}{2\left\{\phi_2^G, T_2\right\}}\ ,\ \ c = -\frac{\left\{-\frac{i}{2}T_1 + bT_2 + r_k\phi_{3,k}^{(1)}\phi_{3,0}^{(1)}, \Phi_1\right\}}{\left\{\phi_2^G, \Phi_1\right\}}\ ,\ \ r_k = \frac{\left(\pi^0 - i\xi^0\right)\zeta qF_{k0}}{4\tilde{\omega}_0}\ .$$

The second pair is $\Phi_8,\ \Phi_9$, where $\Phi_8 = \phi_3^G + d\phi_2^G + v\phi_1^G + u\ \phi_2^{(1)},\quad \Phi_9 = \phi_1^{(1)}$, and

$$d = -\frac{\left\{\phi_3^G, \Phi_1\right\}}{\left\{\phi_2^G, \Phi_1\right\}}\ ,\ \ v = -\frac{\left\{\phi_3^G, \Phi_6\right\}}{\left\{\Phi_7, \Phi_6\right\}}\ ,\ \ u = -\frac{\left\{\phi_3^G + v\phi_1^{(1)}, \phi_4^G\right\}}{\left\{\phi_2^{(1)}, \phi_4^G\right\}}\ .$$

The third pair is $\Phi_{10},\ \Phi_{11}$, where $\Phi_{10} = \phi_4^G + w\phi_2^G + z\Phi_7 + s\Phi_6\ ,\quad \Phi_{11} = \phi_2^{(1)}$, and

$$w = -\frac{\left\{\phi_4^G, \Phi_1\right\}}{\left\{\phi_2^G, \Phi_1\right\}}\ ,\ \ z = -\frac{\left\{\phi_4^G, \Phi_6\right\}}{\left\{\Phi_7, \Phi_6\right\}}\ ,\ \ s = -\frac{\left\{\phi_4^G, \Phi_7\right\}}{\left\{\Phi_6, \Phi_7\right\}}\ .$$

The last pair is $\Phi_{12},\ \Phi_{13}$, where $\Phi_{12} = \phi_{3,0}^{(1)},\quad \Phi_{13} = \phi_{3,4}^{(1)}$. All nonzero Poisson brackets between the new constraints are listed below

$$\begin{aligned}
\left\{\Phi_2, \Phi_1\right\}\ &= -\left\{\Phi_1, \Phi_2\right\} = 1\ ,\ \ \left\{\Phi_3, \Phi_3\right\} = \left\{\Phi_4, \Phi_4\right\} = \left\{\Phi_5, \Phi_5\right\} = -2i\ ,\\
\left\{\Phi_6, \Phi_7\right\}\ &= \left\{\Phi_7, \Phi_6\right\} = \zeta(\tilde{\omega}_0 + m)\ ,\ \ \left\{\Phi_8, \Phi_9\right\} = \left\{\Phi_9, \Phi_8\right\} = 1\ ,\\
\left\{\Phi_{10}, \Phi_{11}\right\}\ &= -\left\{\Phi_{11}, \Phi_{10}\right\} = 1\ ,\ \ \left\{\Phi_{12}, \Phi_{12}\right\} = -\left\{\Phi_{13}, \Phi_{13}\right\} = 2i\ .
\end{aligned} \tag{5.14}$$

Now we are in position to analyze the equations of motion in the case under consideration. They have the form (2.14), in which one has to put $H = 0$,

$$\dot{\eta} = \left\{\eta, \varepsilon\right\}_{D(\Phi)}\ ,\ \ \Phi = 0\ , \tag{5.15}$$

where η stands for the set of all the variables of the theory, and the Dirac brackets are considered in the extended phase space (see Sect. II). We are going to demonstrate that

an effective Hamiltonian exists in this case. To this end let us divide the complete set of constraints Φ into two subsets of constraints, U and V , $\Phi=(U,V)$, where $U=(\Phi_a)$, $a=1,...,5$, $V=(\Phi_b)$, $b=6,...,13$. It is easy to see that both U and V are sets of second-class constraints. In this case the Dirac brackets with respect to the constraints Φ may be calculated successively (see [20], p.276),

$$\{\mathcal{F},\mathcal{G}\}_{D(\Phi)}=\{\mathcal{F},\mathcal{G}\}_{D(U)}-\{\mathcal{F},V_b\}_{D(U)}C^{bd}\{V_d,\mathcal{G}\}_{D(U)}\ , \tag{5.16}$$

where $C^{bd}\{V_d,V_c\}_{D(U)}=\delta_c^b$ and $\mathcal{F},\ \mathcal{G}$ are some functions on phase variables. Consider only variables $\boldsymbol{\eta}=\left(x^k,p_k,\zeta,\xi^k,\pi_k\right)$. All other variables may be expressed via these variables, or eliminate from the consideration by means of constraints. Applying the formula (5.16), and taking into account the specific structure of the constraints V, we may write equations of motion for the variables $\boldsymbol{\eta}$ in the following simple form

$$\dot{\boldsymbol{\eta}}=\{\boldsymbol{\eta},\varepsilon\}_{D(U)},\ U=0\ . \tag{5.17}$$

Now let us divide the complete set of constraints U into two subsets of constraints, u and v, $U=(u,v)$, where $u=(\Phi_a)$, $a=3,4,5$, $v=(\Phi_b)$, $b=1,2$. It is easy to see that both u and v are sets of second-class constraints. Now we may again calculate the Dirac brackets from Eq. (5.17) successively. Here a simplification comes from the fact that $\{\boldsymbol{\eta},\varepsilon\}_{D(u)}=0$. Thus we get

$$\{\boldsymbol{\eta},\varepsilon\}_{D(U)}=-\{\boldsymbol{\eta},v_a\}_{D(u)}c^{ab}\{v_b,\varepsilon\}_{D(u)}\ ,\quad c^{ab}\{v_b,v_d\}_{D(u)}=\delta_d^a\ . \tag{5.18}$$

The matrix c may be easily calculated: $c^{11}=c^{22}=0,\ c^{12}=-c^{21}=1$. Then the above equation may be written as

$$\{\boldsymbol{\eta},\varepsilon\}_{D(U)}=\{\boldsymbol{\eta},\zeta\Phi_1\}_{D(u)}\ . \tag{5.19}$$

The term $\zeta\Phi_1$ under the Dirac bracket sign in (5.19) may be transformed in the following way: First we may eliminate the momentum p_0 from Φ_1 (there is no x^0 in $\boldsymbol{\eta}$), then substitute x^0 by $\zeta\tau$ according to the constraint $\Phi_2=0$ (there is no p_0 in $\boldsymbol{\eta}$ and in u), and finally to

express all the momenta π_k by $-i\xi_k$ according to the constraints $u=0$ (that may be done since the Dirac brackets are just taken with respect of the constraints u). Thus, finally we may write the equations of motion for the variables $\boldsymbol{\eta}$ in the following form

$$\dot{\boldsymbol{\eta}}=\{\boldsymbol{\eta},\mathcal{H}_{eff}\}_{D(u)}\,,\quad u_k=\phi^{(1)}_{3,k}=0\,,\quad k=1,2,3\ . \tag{5.20}$$

where the effective Hamiltonian H_{eff} reads:

$$\begin{aligned}
&\mathcal{H}_{eff}=[\zeta qA_0+\omega]_{x^0=\zeta\tau}\ ,\quad \omega=\tilde{\omega}|_{\pi_k=-i\xi_k}=\sqrt{\omega_0^2+\rho}\,,\\
&\omega_0=\sqrt{m^2+(p_k+qA_k)^2-2iqF_{kl}\xi^k\xi^l}\,,\quad \rho=\frac{-4i\zeta qF_{k0}}{\omega_0+m}(p_l+qA_l)\xi^k\xi^l\ .
\end{aligned} \tag{5.21}$$

The nonzero Dirac brackets between the independent variables $\boldsymbol{\eta}$ have the form

$$\left\{x^k,p_l\right\}_{D(u)}=\left\{x^k,p_l\right\}=\delta^k_l,\quad \left\{\xi^k,\xi^l\right\}_{D(u)}=\frac{i}{2}\eta^{kl}\ .$$

Then the equal time commutation relations for the operators $\hat{X}^k,\hat{P}_k,\hat{\zeta},\hat{\Xi}^k$, which correspond to the variables x^k,p_k,ζ,ξ^k, we define according to their Dirac brackets. The nonzero commutators (anticommutators) are

$$[\hat{X}^k,\hat{P}_j]=i\hbar\delta^k_j\,,\quad [\hat{\Xi}^k,\hat{\Xi}^l]_+=-\frac{\hbar}{2}\eta^{kl}\ . \tag{5.22}$$

We assume as before $\hat{\zeta}^2=1$, and select a state space $\mathcal{R}$ whose elements $\boldsymbol{\Psi}\in\mathcal{R}$ are $\mathbf{x}$-dependent eight-component columns

$$\boldsymbol{\Psi}=\begin{pmatrix}\Psi_{+1}(\mathbf{x})\\ \Psi_{-1}(\mathbf{x})\end{pmatrix}\,, \tag{5.23}$$

where $\Psi_\zeta(\mathbf{x}),\ \zeta=\pm1$ are four component columns. The inner product in $\mathcal{R}$ is defined as follows:

$$(\boldsymbol{\Psi},\boldsymbol{\Psi}')=\left(\Psi_{+1},\Psi'_{+1}\right)+\left(\Psi'_{-1},\Psi_{-1}\right)\,,\ (\Psi,\Psi')=\int\Psi^\dagger(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x}\ . \tag{5.24}$$

Later on one can see that such a construction of the inner product provides its form invariance under Lorenz transformations. We seek all the operators in the block-diagonal form, in particular, the operators $\hat{\zeta}$ and $\hat{\Xi}^k$ we chose as:

$$\hat{\zeta} = \mathrm{bdiag}\,(I,\ -I)\ ,\ \ \hat{\Xi}^k = \mathrm{bdiag}\left(\hat{\xi}^k,\ \hat{\xi}^k\right)\ ,\ \left[\hat{\xi}^k, \hat{\xi}^l\right]_+ = -\frac{\hbar}{2}\eta^{kl}\ , \tag{5.25}$$

where I is 4×4 unit matrix, and $\hat{\xi}^k$ are some 4×4 matrices, which obey the above equal time commutation relation. Thus, we may realize the operators $\hat{\xi}^k$ by means of γ -matrices,

$$\hat{\xi}^k = \frac{i}{2}\hbar^{1/2}\gamma^k\ ,\ \ \left[\gamma^k, \gamma^l\right]_+ = 2\eta^{kl}\ . \tag{5.26}$$

The canonical operators $\hat{X}^k$ and $\hat{P}_k$ we define as totally diagonal

$$\hat{X}^k = x^k\mathbf{I}\ ,\quad \hat{P}_k = \hat{p}_k\mathbf{I}\ ,\ \ \hat{p}_k = -i\hbar\partial_k\ , \tag{5.27}$$

where $\mathbf{I}$ are 8×8 unit matrix. One can easily see that such defined operators really obey the commutation relations (5.22) and are Hermitian with respect to the inner product (5.24). Evolution of state vectors with the time parameter τ is controlled by a Schrödinger equation with a quantum Hamiltonian $\hat{H}$. The latter may be constructed as a quantum operator in the Hilbert space $\mathcal{R}$ on the base of the correspondence principle starting with its classical analog, which is $\mathcal{H}_{eff}$ given by Eq. (5.21). There exist infinite number of possible operators which have the same classical image. That corresponds to the well-known ambiguity of the quantization in general case. We construct the corresponding quantum Hamiltonian in the following way:

$$\hat{H}(\tau) = \hat{\zeta}q\hat{A}_0 + \hat{\Omega}\ . \tag{5.28}$$

The operator $\hat{A}_0$ has the following diagonal form $\hat{A}_0 = \mathrm{bdiag}\left(\left.A_0\right|_{x^0=\tau} I,\ \ \left.A_0\right|_{x^0=-\tau} I\right)$. We define the operator $\hat{\Omega}$ as follows

$$\hat{\Omega} = \mathrm{bdiag}\left(\left.\hat{\omega}_0\right|_{x^0=\tau},\ -\left.\hat{\omega}_0\right|_{x^0=-\tau}\right)\ ,\ \ \hat{\omega}_0 = \gamma^0\left[m + \gamma^k\left(\hat{p}_k + qA_k\right)\right]\ , \tag{5.29}$$

where γ^0 is one of the Dirac matrix, $\left(\gamma^0\right)^2 = 1,\ \ \left[\gamma^0, \gamma^k\right]_+ = 0$. The first term in the expression (5.28) is a natural quantum image of the classical quantity $\zeta q\left.A_0\right|_{x^0=\zeta\tau}$. Below we are going to adduce some arguments demonstrating that the second term $\hat{\Omega}$ may be considered as a quantum image of the classical quantity $\left.\omega\right|_{x^0=\zeta\tau}$. In fact, we have to justify the following symbolic relation

$$\lim_{classical} \hat{\Omega} = \omega\big|_{x^0=\zeta\tau} \ . \tag{5.30}$$

To be more rigorous one has to work with operator symbols. However, we remain here in terms of operators, hoping that our manipulations have a clear sense and do not need to be confirmed on the symbol language. First, we may replace the operator $\hat{\Omega}$ under the sign of the limit by another one $\hat{\Omega}' = \hat{\Omega} + \hat{\Delta}$, where

$$\hat{\Delta} = \mathrm{bdiag}\left(\gamma^0\hat{\xi}^k\hat{\lambda}_k\Big|_{x^0=\tau},\ \gamma^0\hat{\xi}^k\hat{\lambda}_k\Big|_{x^0=-\tau}\right),\ \ \hat{\lambda}_k = -\hbar^{-1/2}m\left[\left[qF_{l0}\hat{\xi}^l, \frac{1}{\hat{\omega}_0^2 - m^2}\right]_+, \hat{\xi}_k\right].$$

Indeed, one may see that the classical limit of the operator $\hat{\Delta}$ is zero. A justification may be the following: The leading in $\hbar$ contribution to the operator $\hat{\Delta}$ results from the terms, which contain $\left(\hat{\xi}^k\right)^2$. Such operators have classical limit zero. That is related, for example, to the fact that due to the realization (5.26) they are proportional to $\hbar$. On the other side, we may remember that in the classical limit such terms turn out to be proportional to $\left(\xi^k\right)^2$, which is zero due to Grassmann nature of ξ . Both considerations are consistent. As to the operator $\hat{\Omega}'$, we may consider its square and see that its classical limit corresponds to the square of the classical quantity $\omega\big|_{x^0=\zeta\tau}$. It would be enough to prove the relation (5.30). The concrete details look as follows

$$\left(\hat{\Omega}'\right)^2 = \mathrm{bdiag}\left(\hat{\omega}^2\big|_{x^0=\tau,\zeta=1}, \hat{\omega}^2\big|_{x^0=-\tau,\zeta=-1}\right),\ \hat{\omega}^2 = \hat{\omega}_0^2 + \hat{\rho}_1 + \hat{\rho}_2,\ \hat{\omega}_0^2 = \left[m^2 + (\hat{p}_k + qA_k)^2\right] I$$
$$-iqF_{jl}[\hat{\xi}^j, \hat{\xi}^l],\ \hat{\rho}_1 = \frac{1}{2i}\left[\left[\zeta qF_{k0}\hat{\xi}^k, \frac{1}{\hat{\omega}_0^2 - m^2}\right]_+, \left[(\hat{p}_j + qA_j)\,\hat{\xi}^j, \hat{\omega}_0 - m\right]_+\right],$$
$$\hat{\rho}_2 = \frac{m}{2i}\left[\left[\left[\zeta qF_{k0}\hat{\xi}^k, \frac{1}{\hat{\omega}_0^2 - m^2}\right]_+, \hat{\xi}^j\right]_+, (\hat{p}_j + qA_j)\right] - \frac{\hbar\hat{\lambda}_k^2}{4} - \frac{[\hat{\xi}^k, \hat{\xi}^j]}{4}\left[\hat{\lambda}_k + 2\,(\hat{p}_k + qA_k)\,, \hat{\lambda}_j\right].$$

Consideration of the classical limit may be done on the states with a definite value of ζ. One can easily see that in such a limit $\hat{\omega}_0^2 \to \omega_0^2$ and $\hat{\rho}_1 \to \rho$. The operator $\hat{\rho}_2$ is zero in classical limit, since does not contain terms without $\hbar$. Thus, in the classical limit the operator $\left(\hat{\Omega}'\right)^2$, and therefore $\left(\hat{\Omega}\right)^2$ as well, corresponds to the classical quantity $\omega^2|_{x^0=\zeta\tau}$. Returning to our choice of the operator $\hat{\Omega}$, we may say that the classical theory gives complete information about its structure. We have to select nonclassical parts of the operator using additional

considerations. Thus, the form (5.29) was selected to maintain Lorentz invariance of the results of the quantization.

The quantum Hamiltonian (5.28) may be written in the following block-diagonal form convenient for the further consideration,

$$\hat{H}(\tau) = \mathrm{bdiag}\left(\hat{h}(\tau), -\hat{h}(-\tau)\right) \ , \ \ \hat{h}(\tau) = \hat{h}(x^0)\Big|_{x^0=\tau} \ , \ \ \hat{h}(x^0) = qA_0 + \hat{\omega}_0 \ . \tag{5.31}$$

One can see that $\hat{h}(x^0)$ has a form of the one-particle Dirac Hamiltonian.

The states of the system under consideration evolute in time τ in accordance with the Schrödinger equation

$$i\hbar\partial_\tau \mathbf{\Psi}(\tau) = \hat{H}(\tau)\mathbf{\Psi}(\tau) \ , \tag{5.32}$$

where the state vectors $\mathbf{\Psi}$ depend now parametrically on τ,

$$\mathbf{\Psi}(\tau) = \begin{pmatrix} \Psi_{+1}(\tau, \mathbf{x}) \\ \Psi_{-1}(\tau, \mathbf{x}) \end{pmatrix} \ . \tag{5.33}$$

Let us now demonstrate that the equation (5.32) is equivalent to two Dirac equations, one for the Dirac field of the charge q, and another one for the Dirac field of the charge $-q$. In accordance with our classical interpretation we may regard $\hat{\zeta}$ as charge sign operator. Let $\mathbf{\Psi}_\zeta$ be states with a definite charge ζq, thus, $\hat{\zeta}\mathbf{\Psi}_\zeta = \zeta\mathbf{\Psi}_\zeta \ , \ \ \zeta = \pm 1$. It is easily to see that states $\mathbf{\Psi}_{+1}$ with the charge q have $\Psi_{-1} = 0$. In this case $\tau = x^0$, where x^0 is physical time. Then the equation (5.32) may be rewritten as

$$i\hbar\partial_0 \Psi_{+1}(x^0, \mathbf{x}) = \hat{h}(x^0)\Psi_{+1}(x^0, \mathbf{x}) \ .$$

Denoting $\Psi_{+1}(x^0, \mathbf{x}) = \psi(x)$ we get exactly the Dirac equation for the spinor field $\psi(x)$ with charge q,

$$\left[\gamma^\mu \left(i\hbar\partial_\mu - qA_\mu\right) - m\right]\psi(x) = 0. \tag{5.34}$$

States $\mathbf{\Psi}_{-1}$ with charge $-q$ have $\Psi_{+1} = 0$. In this case, according to our classical interpretation, $\tau = -x^0$, where x^0 is physical time. Using, for example, the standard representation of the Dirac matrices, one can see that

$$\hat{h}^c(x^0) = \gamma^2 \left(\hat{h}(x^0)\right)^* \gamma^2 = \hat{h}(x^0)\Big|_{q\to -q} , \tag{5.35}$$

where $\hat{h}^c(x^0)$ is the charge conjugated Dirac Hamiltonian. Then, we get from the equation (5.32)

$$i\hbar\partial_0 \Psi^*_{-1}(-x^0, \mathbf{x}) = -\gamma^2 \hat{h}^c(x^0)\gamma^2 \Psi^*_{-1}(-x^0, \mathbf{x}) .$$

Denoting $\gamma^2\Psi^*_{-1}(-x^0, \mathbf{x}) = \psi^c(x)$ one may rewrite this equation in the form of the Dirac equation for the charge conjugated spinor field $\psi^c(x)$ (that which describes particles with the charge $-q$),

$$\left[\gamma^\mu \left(i\hbar\partial_\mu + qA_\mu\right) - m\right]\psi^c(x) = 0. \tag{5.36}$$

The inner product (5.24) between two solutions of the Schrödinger equation (5.32) with different charges is zero. For two solutions with charges q it takes the form:

$$\left(\boldsymbol{\Psi}_{+1}, \boldsymbol{\Psi}'_{+1}\right) = \int \psi^+(x)\psi(x)d\mathbf{x} = (\psi, \psi')_D ,$$

and is expressed via the Dirac scalar product on the $x^0 = \mathrm{const}$ hyperplane for the case of the charge q. For two solutions with charges $-q$ the inner product (5.24) reads:

$$\left(\boldsymbol{\Psi}_{-1}, \boldsymbol{\Psi}'_{-1}\right) = \int \psi^{c+}(x)\psi^{c'}(x)d\mathbf{x} = (\psi^c, \psi^{c'})_D ,$$

and is expressed via the Dirac scalar product for the case of the charge $-q$.

Let us study the eigenvalue problem for the Dirac Hamiltonian (5.31) in a time independent external backgrounds (thus, below this Hamiltonian does not depend on x^0):

$$\hat{h}\psi(\mathbf{x}) = \epsilon\psi(\mathbf{x}) . \tag{5.37}$$

Here ϵ defines the energy spectrum of particles with the charge q. As usual, it is convenient to present the Dirac spinor in the form

$$\psi(\mathbf{x}) = \left[\gamma^0 \left(\epsilon - qA_0\right) + \gamma^k \left(i\hbar\partial_k - qA_k\right) + m\right]\varphi(\mathbf{x}) .$$

Then the function $\varphi(\mathbf{x})$ obeys the squared Dirac equation,

$$
\left[(\epsilon - qA_0)^2 - D\right]\varphi(\mathbf{x}) = 0\,,\;\; D = m^2 + (i\hbar\partial_k - qA_k)^2 + \frac{i}{4}qF_{\mu\nu}[\gamma^\mu,\gamma^\nu]_- \,. \tag{5.38}
$$

The main features of such a spectrum in general case (for non-superstrong potentials A_0) may be derived from the equation (5.38) repeating the discussion presented for the scalar field. First of all, one may see that a pair $(\varphi,\ \epsilon)$ is a solution of the equation (5.38) if it obeys either the equation $\epsilon = qA_0 + \sqrt{\varphi^{-1}D\varphi}$, or the equation $\epsilon = qA_0 - \sqrt{\varphi^{-1}D\varphi}$. Let us denote via $(\varphi_{+,n},\ \epsilon_{+,n})$ solutions of the first equation, and via $(\varphi_{-,\alpha},\ \epsilon_{-,\alpha})$ solutions of the second equation, where n and α are some quantum numbers which are different in general case. Thus,

$$
\epsilon_{+,n} = qA_0 + \sqrt{\varphi_{+,n}^{-1}D\varphi_{+,n}}\,,\quad \epsilon_{-,\alpha} = qA_0 - \sqrt{\varphi_{-,\alpha}^{-1}D\varphi_{-,\alpha}}\,. \tag{5.39}
$$

One can call $\epsilon_{+,n}$ the upper branch of the energy spectrum and $\epsilon_{-,\alpha}$ the lower branch of the energy spectrum. We get $(\psi_{+,n},\ \epsilon_{+,n})$ and $(\psi_{-,\alpha},\ \epsilon_{-,\alpha})$ solutions of the eigenvalue problem (5.37), where

$$
\begin{aligned}
\psi_{+,n}(\mathbf{x}) &= \left[\gamma^0\left(\epsilon_{+,n} - qA_0\right) + \gamma^j\left(i\hbar\partial_j - qA_j\right) + m\right]\varphi_{+,n}(\mathbf{x})\,,\\
\psi_{-,n}(\mathbf{x}) &= \left[\gamma^0\left(\epsilon_{-,n} - qA_0\right) + \gamma^j\left(i\hbar\partial_j - qA_j\right) + m\right]\varphi_{-,n}(\mathbf{x})\,.
\end{aligned} \tag{5.40}
$$

Square of the Dirac norm of the eigenvectors $\psi_{\varkappa,n}$ is positive and they may be orthonormalized as follows,

$$
\left(\psi_{\varkappa,n},\psi_{\varkappa',n'}\right)_D = \delta_{\varkappa,\varkappa'}\delta_{n,n'},\quad \varkappa = \pm\,. \tag{5.41}
$$

A solution of the eigenvalue problem for the charge conjugated Dirac Hamiltonian $\hat{h}^c$,

$$
\hat{h}^c\psi^c_{\varkappa,n} = \epsilon^c_{\varkappa,n}\psi^c_{\varkappa,n}\,, \tag{5.42}
$$

one can find using Eq. (5.35). Then

$$
\psi^c_{\varkappa,n} = \gamma^2\psi^*_{-\varkappa,n}\,,\;\; \epsilon^c_{\varkappa,n} = -\epsilon_{-\varkappa,n}\,,\; \left(\psi^c_{\varkappa,n},\psi^c_{\varkappa',n'}\right) = \delta_{\varkappa,\varkappa'}\delta_{n,n'},\;\; \varkappa = \pm\,. \tag{5.43}
$$

Proceeding similar to the scalar particle case in x^0-representation, we define orthogonal each other sets $\mathbf{\Psi}_{+,n}$, and $\mathbf{\Psi}^c_{+,\alpha}$,

$$\hat{H}\boldsymbol{\Psi}_{+,n} = \epsilon_{+,n}\boldsymbol{\Psi}_{+,n}, \quad \boldsymbol{\Psi}_{+,n} = \begin{pmatrix} \psi_{+,n}(\mathbf{x}) \\ 0 \end{pmatrix}, \; (\boldsymbol{\Psi}_{+,n}, \boldsymbol{\Psi}_{+,m}) = \delta_{nm} ,$$

$$\hat{H}\boldsymbol{\Psi}^c_{+,\alpha} = \epsilon^c_{+,\alpha}\boldsymbol{\Psi}^c_{+,\alpha}, \quad \boldsymbol{\Psi}^c_{+,\alpha} = \begin{pmatrix} 0 \\ \psi^c_{+,\alpha}(\mathbf{x}) \end{pmatrix}, \; \left(\boldsymbol{\Psi}^c_{+,\alpha}, \boldsymbol{\Psi}^c_{+,\beta}\right) = \delta_{\alpha\beta} ,$$

where $\hat{H} = \mathrm{bdiag}\left(\hat{h},\ \hat{h}^c\right)$. The sets form a complete basis in the physical subspace $\mathcal{R}^1_{ph}$. Now we can see that $\hat{\Omega}$ is positive defined in the physical subspace in accordance with positivity of classical value ω. That positivity condition helps to fix an ambiguity in the definition of $\hat{\Omega}$. For example, in the τ-representation we could define the operator $\hat{\Omega}$ as follows, $\hat{\Omega} = \mathrm{bdiag}\left(\hat{\omega}_0|_{x^0=\tau} ,\ \pm\,\hat{\omega}_0|_{x^0=-\tau}\right)$. We need the positivity condition to select the minus sign in the lower block, as was done in (5.29).

To complete the consideration, as in spinless case, we examine the Dirac quantization of the theory in question. In this case we do not need to impose any gauge condition to the first-class constraints. We assume as before the operator $\hat{\zeta}$ to have the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory. The equal time commutation relations for the operators $\hat{X}^\mu, \hat{P}_\mu, \hat{\zeta}, \hat{\Xi}^n, \hat{e}, \hat{P}_e, \hat{\chi}, \hat{P}_\chi$ which correspond to the variables $x^\mu, p_\mu, \zeta, \xi^n, e, P_e, \chi, P_\chi$, we define according to their Dirac brackets, with respect to second-class constraints $\phi^{(1)}_{3,n}$. Thus, now we get

$$[\hat{X}^\mu, \hat{P}_\nu]_- = i\hbar\delta^\mu_\nu \ , \ \ [\hat{\Xi}^n, \hat{\Xi}^m]_+ = -\frac{\hbar}{2}\eta^{nm} \ , \ \ \hat{\zeta}^2 = 1 \ , \ \ \left[\hat{e}, \hat{P}_e\right] = i\hbar \, , \ \ \left[\hat{\chi}, \hat{P}_\chi\right]_+ = i\hbar \ , \tag{5.44}$$

whereas all other commutators (anticommutators) equals zero. Besides, we have to keep in mind the necessity to construct an operator realization for the first class-constraint T_2 from (5.7), which contains a square root. Taking all that into account, we select as a state space one whose elements $\boldsymbol{\Psi}$ are x-dependent eight-component columns

$$\boldsymbol{\Psi} = \begin{pmatrix} \Psi_{+1}(x) \\ \Psi_{-1}(x) \end{pmatrix} , \tag{5.45}$$

where $\Psi_\zeta(x),\ \zeta = \pm 1$ are four-component columns. We seek all the operators in the block-diagonal form,

$$\hat{\zeta}=\mathrm{bdiag}\,(I,\ -I)\,,\ \ \hat{X}^{\mu}=x^{\mu}\mathbf{I}\,,\ \ \ \hat{P}_{\mu}=\hat{p}_{\mu}\mathbf{I}\,,\ \ \hat{p}_{\mu}=-i\hbar\partial_{\mu}\,,\ \hat{e}=e\mathbf{I}\,,\,\hat{P}_{e}=-i\hbar\partial_{e}\,\mathbf{I},$$

$$\hat{\Xi}^{n}=\mathrm{bdiag}\left(\hat{\xi}^{n},\hat{\xi}^{n}\right),\ \hat{\xi}^{\mu}=\gamma^{\mu}\hat{\xi}^{4},\ \hat{\xi}^{4}=\frac{i\hbar^{1/2}}{2}\gamma^{5},\ \hat{\chi}=\chi\mathbf{I},\ \hat{P}_{\chi}=-i\hbar\partial_{\chi}\mathbf{I}, \tag{5.46}$$

where I and $\mathbf{I}$ are 4×4 and 8×8 unit matrices respectively, and $\gamma^{5}=\gamma^{0}...\gamma^{3}$.The operator $\hat{T}_1$, which corresponds to the first-class constraint T_1 from (5.6), is selected as $\hat{T}_1=\hat{\Xi}^{\mu}\left(\hat{P}_{\mu}+q\hat{A}_{\mu}\right)+m\hat{\Xi}^{4}$, where $\hat{A}_{\mu}=A_{\mu}\mathbf{I}$. The operator $\hat{T}_2$, which corresponds to the first-class constraint T_2 from (5.7), is selected as $\hat{T}_2=\hat{P}_0+q\hat{A}_0+\hat{\zeta}\hat{R}$, where $\hat{R}=\mathrm{bdiag}\,(\hat{\omega}_0,\ -\hat{\omega}_0)$, $\hat{\omega}_0=\gamma^{0}\left[m+\gamma^{k}\left(\hat{p}_k+qA_k\right)\right]$. Similar to the canonical quantization case, one may verify that the square $\hat{R}^2$ corresponds (in the classical limit) to the square of the classical quantity r, see the end of the Section. The state vectors (5.45) do not depend on "time" τ since the Hamiltonian vanishes on the constraints surface. The physical state vectors have to obey the equations $\hat{P}_e\mathbf{\Psi}=0,\ \hat{P}_{\chi}\mathbf{\Psi}=0,\ \hat{T}_1\mathbf{\Psi}=0,\ \hat{T}_2\mathbf{\Psi}=0.$ First two of these conditions mean that the state vectors do not depend on e and χ. Due to the bloc-diagonal form of the operators $\hat{T}$ the second two conditions produce the following equations for the four-column $\Psi_{\zeta}(x)$,

$$\hat{t}_1\Psi_{\zeta}(x)=0\,,\ \hat{t}_2\Psi_{\zeta}(x)=0\ ,\ \ \zeta=\pm 1\,, \tag{5.47}$$

where $\hat{t}_1=\frac{i}{2}\hbar^{1/2}\gamma^{0}\ \hat{t}_2\gamma^{5}$, $\hat{t}_2=\hat{p}_0+qA_0+\hat{\omega}_0$. These equations are consistent, the first one is a consequence of the second one. Thus, we have in fact one equation, which may be written as follows:

$$\gamma^{0}\hat{t}_2\Psi_{\zeta}(x)=\left[\gamma^{\mu}\left(i\hbar\partial_{\mu}-qA_{\mu}\right)-m\right]\Psi_{\zeta}(x)=0\ ,\ \ \zeta=\pm 1\ . \tag{5.48}$$

Denoting $\Psi_{+1}(x)=\psi(x)$, and $\gamma^{2}\Psi^{*}_{-1}(x)=\psi^{c}(x)$, we get two Dirac equations, one (5.34) for the charge q and another one (5.36) for the charge $-q$.

Finally let us verify that $\lim_{classical}\hat{R}=r$. Under the limit sign we may replace the operator $\hat{R}$ by another one $\hat{R}'=\hat{R}+\hat{\Delta}$, using the same kind of arguments, which were used in canonical quantization case,

$$\hat{\Delta}=\mathrm{bdiag}\left(-\gamma^{k}\gamma^{5}\hat{\xi}^{0}\hat{\lambda}_k,\ \gamma^{k}\gamma^{5}\hat{\xi}^{0}\hat{\lambda}_k\right)\ ,\ \ \hat{\lambda}_k=-4\hbar^{-1/2}\left[qF_{l0}\hat{\xi}^{l}\hat{\xi}^{0},\frac{1}{\hat{p}_k+qA_k}\right]_{+}\ .$$

Then $\left(\hat{R}'\right)^2 = \mathrm{bdiag}\left(\hat{r}^2,\ \hat{r}^2\right)$, $\hat{r}^2 = \hat{\omega}_0^2 + \hat{\rho}_1 + \hat{\rho}_2$, where

$$\hat{\rho}_1 = \frac{\hbar^{1/2}}{2i}\left[\left(\hat{p}_k + qA_k\right), \hat{\lambda}^k\right]_+ ,\ \hat{\rho}_2 = -\frac{\hbar\hat{\lambda}_k^2}{4} - \frac{[\hat{\xi}^k, \hat{\xi}^j]}{4}\left[\hat{\lambda}_k, \hat{\lambda}_j - 4i\hbar^{-1/2}\left(\hat{p}_j + qA_j\right)\right].$$

One can easily see that in classical limit $\hat{\omega}_0^2 \to \omega_0^2$, $\hat{\rho}_1 \to -4iqF_{l0}\,\xi^l\xi^0$, $\hat{\rho}_2 \to 0$. Thus, in the classical limit the operator $\hat{r}^2$, corresponds to the classical quantity $r^2|_{\pi_\mu=-i\xi_\mu} = m^2 + \left(p_k + qA_k\right)^2 - 2iqF_{\mu\nu}\xi^\mu\xi^\nu$.

Unfortunately, the Dirac method of the quantization gives no more information how to proceed further with the consistent quantum theory construction, and moreover contains principal contradictions, see discussion in the Introduction. However, we may conclude that at least one of the main feature of the quantum theory, its charge conjugation invariance, remains also in the frame of the Dirac quantization.

VI. CONCLUDING REMARKS

Thus, we see that the first quantization of classical actions of spinless and spinning particles leads to relativistic quantum mechanics which are consistent to the same extent as corresponding quantum field theories in one-particle sectors. Such quantum mechanics describe the corresponding charged particles of both signs, and reproduce correctly their energy spectra without infinite number of negative energy levels. No negative vector norms need to be used in the corresponding Hilbert spaces.

Certainly, the relativistic quantum mechanics may not be formulated literally in the same terms as a non-relativistic quantum mechanics. For example, there is a problem with position and momentum operator definitions. If one selects as such operators expressions defined by the equations (3.5), then such operators lead state vectors out of the physical subspace. One cannot define a positively defined probability density. All that is a reflection of a well-known fact that it is not possible to construct one-particle localized states in the relativistic theory. It does not depend on the background under consideration. The problem with the momentum operator depends on the external background, and does not exist in

translationary invariant backgrounds.

In backgrounds which violate the vacuum stability of the QFT, a more complicated multi-particle interpretation of the quantum mechanics constructed is also possible, which establish a connection to the QFT. Such an interpretation will be presented in a separate publication.

Finally one ought to discuss a relation between the present results and quantization procedure proposed earlier by Gitman and Tyutin (GT) in the papers [10,11]. As was already mentioned in the Introduction the quantization there was done only for restricted classes of external electromagnetic backgrounds, namely for constant magnetic field. All following attempts (see [12]) to go beyond that type of backgrounds met serious difficulties, which are not accidental. It was not demonstrated that the quantum mechanics constructed in course of the quantization is completely equivalent to the one-particle sector of the QFT. In particular, one may see that quantum version of spinless particle model does not provide right transformation properties of mean values. The principal difference between the present approach to the quantization of RP and the previous one is in a different understanding of the role of the variable ζ. In the papers of GT and in the following papers, which used the same approach, they used this variable to get both branches of solutions of Klein-Gordon equation. In course of a more deep consideration it became clear that this aim can be achieved without the use of this variable. One may select a special realization of the commutation relations in the Hilbert space to get complete Klein-Gordon equation (see Sect.III). Doing such a realization we may naturally include into the consideration arbitrary electromagnetic and even gravitational backgrounds. Nevertheless, the role of the variable ζ turned out to be decisive to reproduce a consistent relativistic quantum mechanics and provide perfect equivalence with the one-particle sector of the QFT. Due to the existence of the variable ζ we double the Hilbert space to describe particles and antiparticles on the same footing. Thus, we solve the problem of negative norms and infinite number of negative energy levels. The existence of the variable ζ makes the first and the second quantizations completely equivalent within the one-particle sector (in cases when it may be

defined consistently). In both cases we start with an action with a given charge q, and in course of the quantization we arrive to theories which describe particles of both charges $\pm q$, and are C -invariant. In case of the first quantization this is achieved due to the existence and due to right treating of the variable ζ. One ought also to remark that the requirement to maintain all classical symmetries under the coordinate transformations and under $U(1)$ transformations allows one to realize operator algebra without any ambiguities.

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APPENDIX A: QUANTUM SCALAR FIELD IN EXTERNAL BACKGROUNDS

1. Classical scalar field

Consider here the theory of complex (charged) scalar field $\varphi(x)$ placed in a curved space-time[7] $(g_{\mu\nu}(x))$ and interacting with an external electromagnetic background, described by potentials $A_\mu(x)$. A corresponding action for such a field theory may be written in the following form[8]:

$$S_{FT} = \int \mathcal{L} dx \;, \;\; \mathcal{L} = \sqrt{-g}\left[(P_\mu\varphi)^* g^{\mu\nu} P_\nu\varphi - m^2\varphi^*\varphi\right] , \;\; P_\mu = i\partial_\mu - qA_\mu \;. \tag{A1}$$

The corresponding Euler-Lagrange equation is covariant Klein-Gordon equation in the background under consideration,

$$\left[\frac{1}{\sqrt{-g}} P_\mu \sqrt{-g} g^{\mu\nu} P_\nu - m^2\right] \varphi(x) = 0 \;. \tag{A2}$$

---

[7] As before we use the gauge $g_{0i} = 0,\; g^{00} = g_{00}^{-1} > 0,\; g^{ik} g_{kj} = \delta^i_j$

[8] In this section we select $\hbar = c = 1$

The Klein-Gordon equation was proposed by a number of authors [39–42]. It was shown that in a certain sense this equation may describe particles with spin zero and the charges $(q, 0, -q)$. However, a corresponding one-particle quantum mechanics, which was constructed to support this interpretation, contains indefinite metric and negative energy spectrum for the positron (antiparticle) branch [1–3,5,6]. A one-particle quantum mechanics of particles with spin one-half and the charges $(q, 0, -q)$, which was constructed on the base of the Dirac equation [43], did not contain indefinite metric but still cannot avoid the negative energy spectrum for antiparticles.

The metric energy momentum tensor and the current density vector calculated from the action (A1) have the form

$$T_{\mu\nu} = (P_\mu\varphi)^* P_\nu\varphi + (P_\nu\varphi)^* P_\mu\varphi - \frac{g_{\mu\nu}}{\sqrt{-g}}\mathcal{L}\,,\ J_\mu = q\left[(P_\mu\varphi)^*\varphi + \varphi^*(P_\mu\varphi)\right]\ . \tag{A3}$$

The latter obeys the continuity equation, which may be written as

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}J_\nu\right) = 0 \to \partial_0\rho + \mathrm{div}\mathbf{j} = 0,\ \rho = J_0 g^{00}\sqrt{-g},\ \mathbf{j} = (j^i),\ j^i = g^{ik}J_k\sqrt{-g}. \tag{A4}$$

Introducing the canonical momenta to the fields $\varphi,\ \varphi^*$,

$$\begin{aligned}
\Pi &= \frac{\partial\mathcal{L}}{\partial\varphi_{,0}} = i\left(P_0\varphi\right)^* g^{00}\sqrt{-g},\ \ \varphi^*_{,0} = \frac{g_{00}\Pi}{\sqrt{-g}} + iqA_0\varphi^*\ ,\\
\Pi^* &= \frac{\partial\mathcal{L}}{\partial\varphi^*_{,0}} = -i\left(P_0\varphi\right) g^{00}\sqrt{-g},\ \ \varphi_{,0} = \frac{g_{00}\Pi^*}{\sqrt{-g}} - iqA_0\varphi\ ,
\end{aligned} \tag{A5}$$

one may pass to Hamiltonian formulation. Calculating in this formulation Hamiltonian, momentum, and electric charge, we get on the $x^0 = \mathrm{const}$ hyperplane

$$\begin{aligned}
&H^{FT}(x^0) = \int T_{00}g^{00}\sqrt{-g}d\mathbf{x}\\
&= \int\left[\frac{g_{00}}{\sqrt{-g}}\Pi^*\Pi - \sqrt{-g}\left(P_k\varphi\right)^* g^{kj}P_j\varphi + 2qA_0\mathrm{Im}(\Pi\varphi) + \sqrt{-g}m^2\varphi^*\varphi\right]d\mathbf{x}\ ,
\end{aligned} \tag{A6}$$

$$P_i^{FT}(x^0) = \int T_{i0}g^{00}\sqrt{-g}d\mathbf{x} = \int 2\mathrm{Im}(\Pi P_i\varphi)d\mathbf{x}\ , \tag{A7}$$

$$Q^{FT} = \int J_0 g^{00}\sqrt{-g}d\mathbf{x} = q\int\left[(P_0\varphi)^*\varphi + \varphi^*(P_0\varphi)\right]g^{00}\sqrt{-g}d\mathbf{x} = q\int 2\mathrm{Im}(\Pi\varphi)d\mathbf{x}\ . \tag{A8}$$

The charge (A8) does not depend on the time x^0 due to equations of motion. Using that fact, one may introduce a conserved inner product of two solutions of the Klein-Gordon equation

$$(\varphi, \varphi')_{KG} = \int \left[(P_0\varphi)^* \varphi' + \varphi^* (P_0\varphi') \right] g^{00}\sqrt{-g} d\mathbf{x} \,. \tag{A9}$$

2. Hamiltonian form of Klein-Gordon equation

The Klein-Gordon equation (A2) may be rewritten in the form of a first order in time equation (Hamiltonian form), which may be interpreted as a Schrödinger equation. That can be done in different ways. For example, let us separate the time derivative part in (A2) from the spatial one,

$$i\partial_0 \left(\sqrt{-g} g^{00} P_0 \varphi \right) = \left[-P_k \sqrt{-g} g^{kj} P_j + m^2 \sqrt{-g} \right] \varphi + qA_0 \left(\sqrt{-g} g^{00} P_0 \varphi \right) \,. \tag{A10}$$

Then it is easy to see that in terms of the columns

$$\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} , \;\; \chi = \sqrt{-g} g^{00} P_0 \varphi = i\Pi^* \tag{A11}$$

the equation (A10) takes the form of the Schrödinger equation

$$i\partial_0 \psi = \hat{h}(x^0)\psi \,, \tag{A12}$$

with the Hamiltonian

$$\hat{h}(x^0) = \hat{\omega} + qA_0, \;\; \hat{\omega} = \begin{pmatrix} 0 & M \\ G & 0 \end{pmatrix} , \; G = \frac{g_{00}}{\sqrt{-g}} \,, \tag{A13}$$
$$M = -P_k \sqrt{-g} g^{kj} P_j + m^2 \sqrt{-g} = -\left[\hat{p}_k + qA_k \right] \sqrt{-g} g^{kj} \left[\hat{p}_j + qA_j \right] + m^2 \sqrt{-g}, \; \hat{p}_k = -i\partial_k \,.$$

One can express the Hamiltonian (A6), the momentum (A7), and the charge (A8) in terms of the columns (A11),

$$H^{FT}(x^0) = \int \overline{\psi} \hat{h}(x^0) \psi d\mathbf{x} \,, \;\; P_i^{FT}(x^0) = \int \overline{\psi} P_i \psi d\mathbf{x} \,, \;\; Q^{FT} = q \int \overline{\psi} \psi d\mathbf{x}, \;\; \overline{\psi} = \psi^+ \sigma_1 \,. \tag{A14}$$

The continuity equation follows from (A12) and may be written as

$$\partial_0\rho + \mathrm{div}\mathbf{j} = 0,\ \rho = \overline{\psi}\psi,\ j^i = \frac{1}{2}g^{ik}\sqrt{-g}\left[\overline{\psi}P_k + \overline{(P_k\psi)}\right](\sigma_1 + i\sigma_2)\psi\ . \tag{A15}$$

The Klein-Gordon inner product (A9) takes then the form

$$(\varphi,\varphi')_{KG} = (\psi,\psi') = \int \overline{\psi}(x)\psi'(x)d\mathbf{x} = \int \left[\chi^*(x)\varphi'(x) + \varphi^*(x)\chi'(x)\right]d\mathbf{x}\ . \tag{A16}$$

It is easy to see that the Hamiltonian $\hat{h}(x^0)$ is Hermitian with respect to the inner product (A16).

One may say that a set $\psi_B(x)$ (where B are some quantum numbers) of solutions of the Klein-Gordon equation (A12) is complete if any solution of this equation may be decomposed via the set. If this set is orthogonal with respect to the inner product (A16), then the completeness relation may be written in the following form

$$\sum_B \left.\frac{\psi_B(x)\overline{\psi}_B(y)}{(\psi_B,\psi_B)}\right|_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y})\ . \tag{A17}$$

In terms of the scalar component φ (see (A11) this condition reads

$$\sum_B \left.\frac{\varphi_B(x)\varphi_B^*(y)}{(\varphi_B,\varphi_B)_{KG}}\right|_{x^0=y^0} = 0\,,\ \sum_B \left.\frac{\varphi_B(x)\sqrt{-g(y)}g^{00}(y)\left(P_0\varphi_B(y)\right)^*}{(\varphi_B,\varphi_B)_{KG}}\right|_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y})\ . \tag{A18}$$

The Klein-Gordon equation in the common second order form (A2) is invariant under the operation $q \to -q,\ \varphi \to \varphi^c = \varphi^*$, which is in fact the charge conjugation operation. That means that if $\varphi(x)$ is a wave function of a scalar particle with a charge q then $\varphi^c(x)$ is a wave function for that with the charge $-q$. For the Klein-Gordon equation in the first order form (A12) such an operation looks a little bit more complicated. Using the relation (3.16), one may see that the Klein-Gordon equation in the first order form (A12) is invariant under the following operation $q \to -q,\ \psi \to \psi^c = -\sigma_3\psi^*$, so that

$$i\partial_0\psi^c = \hat{h}^c(x^0)\psi^c,\ \ \hat{h}^c(x^0) = \left.\hat{h}(x^0)\right|_{q\to -q} = -\left[\sigma_3\hat{h}(x^0)\sigma_3\right]^*\ . \tag{A19}$$

Thus defined charge conjugation for two columns (A11) is matched with the charge conjugation for scalar wave functions.

3. Solutions and spectrum of Klein-Gordon equation

Let us study the eigenvalue problem for the Hamiltonian (A13) in time independent external backgrounds (thus, below this Hamiltonian does not depend on x^0):

$$\hat{h}\psi(\mathbf{x}) = \epsilon\psi(\mathbf{x}) \ , \ \ \psi(\mathbf{x}) = \begin{pmatrix} \chi(\mathbf{x}) \\ \varphi(\mathbf{x}) \end{pmatrix} \ . \tag{A20}$$

Being written in components, the equation (A20) takes the form:

$$qA_0\chi + M\varphi = \epsilon\chi \, , \ \ qA_0\varphi + G\chi = \epsilon\varphi \ . \tag{A21}$$

The system (A21) results in the following equation for φ:

$$GM\varphi = [\epsilon - qA_0]^2\varphi \ \Rightarrow \left[(\epsilon - qA_0)^2 g^{00} + \frac{1}{\sqrt{-g}} P_k \sqrt{-g} g^{kj} P_j - m^2 \right] \varphi = 0 \ . \tag{A22}$$

If we make the substitution $\varphi(x) = \exp[-i\epsilon x^0]\varphi(\mathbf{x})$ in the Klein-Gordon equation (A2) we arrive just to the equation (A22). Thus, ϵ defines the energy spectrum of the Klein-Gordon equation for particles with the charge q. Such a spectrum is well known for free background and for special exact solvable cases of external electromagnetic and gravitational fields [29,44,32,28,30]. The main features of such a spectrum in general case (for non-superstrong potentials A^0) may be derived from the equation (A22). First of all, one may see that a pair $(\varphi, \ \epsilon)$ is a solution of the equation (A22) if it obeys either the equation $\epsilon = qA_0 + \sqrt{\varphi^{-1}GM\varphi}$, or the equation $\epsilon = qA_0 - \sqrt{\varphi^{-1}GM\varphi}$. Let us denote via $(\varphi_{+,n}, \ \epsilon_{+,n})$ solutions of the first equation, and via $(\varphi_{-,\alpha}, \ \epsilon_{-,\alpha})$ solutions of the second equation, where n and α are some quantum numbers which are different in general case. Thus,

$$\epsilon_{+,n} = qA_0 + \sqrt{\varphi_{+,n}^{-1} GM \varphi_{+,n}} \ , \ \ \epsilon_{-,\alpha} = qA_0 - \sqrt{\varphi_{-,\alpha}^{-1} GM \varphi_{-,\alpha}} \ . \tag{A23}$$

It is clear that

$$\epsilon_{+,n} - \epsilon_{-,\alpha} = \sqrt{\varphi_{+,n}^{-1} GM \varphi_{+,n}} + \sqrt{\varphi_{-,\alpha}^{-1} GM \varphi_{-,\alpha}} \ > 0 \ . \tag{A24}$$

Thus, one can call $\epsilon_{+,n}$ the upper branch of the energy spectrum and $\epsilon_{-,\alpha}$ the lower branch of the energy spectrum. In the presence of the potential A_0 they may be essentially nonsymmetric, as an example one can remember the energy spectrum in Coulomb field, where (for an attractive Coulomb potential for the charge q) the upper branch contains both discrete and continuous parts of energy levels and the lower branch contains only continuous levels, see Fig.1.

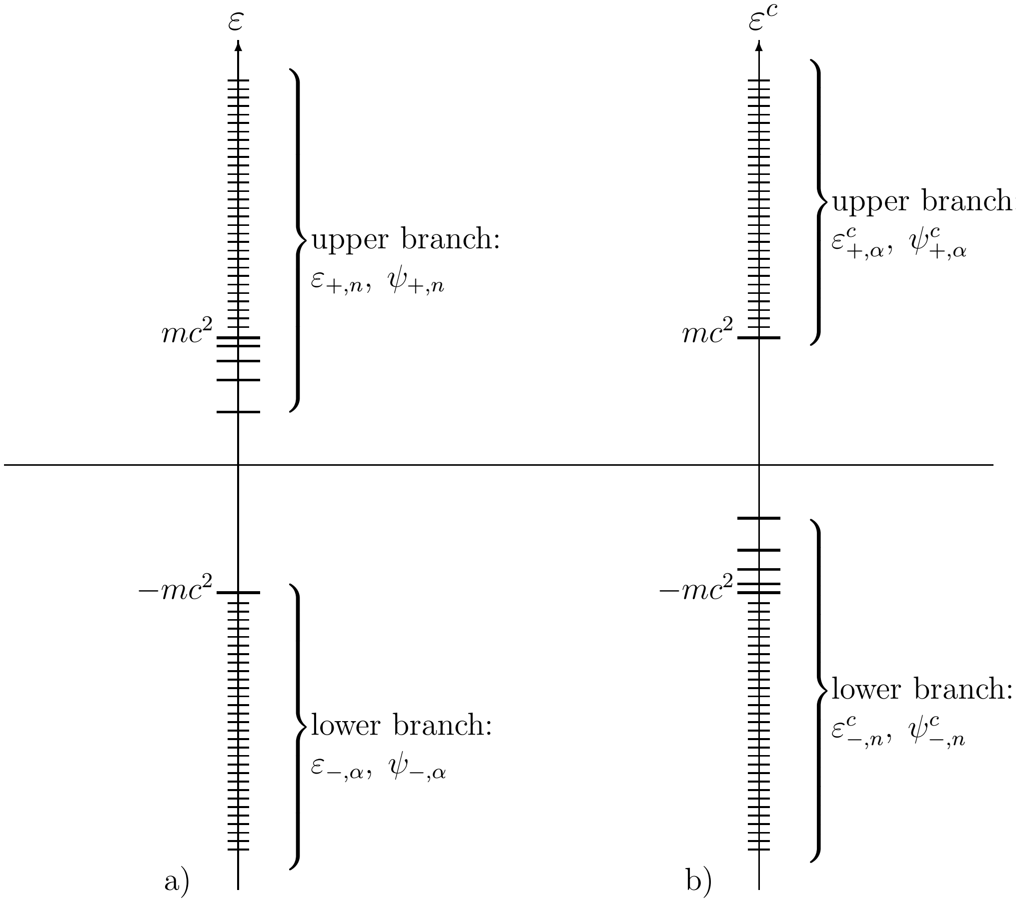

Fig.1. Energy spectra of Klein-Gordon particles with a charge q and $-q$; a) - spectrum of $\hat{h}$, b) - spectrum of $\hat{h}^c$.

In the absence of the potential A_0, one can always select equal quantum numbers for both branches, thus, the total spectrum becomes symmetric,

$$\epsilon_{\pm,n} = \pm\sqrt{\varphi^{-1}_{\pm,n} GM\varphi}_{\pm,n} = \mp\,\epsilon_{\mp,n}\ .$$

Further, even in general case when A_0 is not zero, we are going to use sometimes the same

index n to label quantum numbers both for upper and lower branches to simplify equations, hoping that it does not lead to a misunderstanding for those readers who keeps in mind the above explanations.

We may express the functions χ from the equation (A21),

$$\chi_{\varkappa,n} = g^{00}\sqrt{-g}\left(\epsilon_{\varkappa,n} - qA_0\right)\varphi_{\varkappa,n}, \quad \varkappa = \pm \, .$$

Thus,

$$\hat{h}\psi_{\varkappa,n} = \epsilon_{\varkappa,n}\psi_{\varkappa,n}, \; \psi_{\varkappa,n} = \begin{pmatrix} g^{00}\sqrt{-g}\left(\epsilon_{\varkappa,n} - qA_0\right)\varphi_{\varkappa,n} \\ \varphi_{\varkappa,n} \end{pmatrix} . \tag{A25}$$

Calculating square of the norm of the eigenvectors $\psi_{\varkappa,n}$, using the inner product (A16), we find

$$\left(\psi_{\varkappa,n}, \psi_{\varkappa,n}\right) = 2\int \left(\epsilon_{\varkappa,n} - qA_0\right)\left|\varphi_{\varkappa,n}\right|^2 g^{00}\sqrt{-g}d\mathbf{x} \, . \tag{A26}$$

Taking into account the positivity of g^{00} and the relation $\operatorname{sign}\left(\epsilon_{\varkappa,n} - qA_0\right) = \varkappa$, which follows from (A23), we can see that $\operatorname{sign}\left(\psi_{\varkappa,n}, \psi_{\varkappa,n}\right) = \varkappa$. Since the Hamiltonian $\hat{h}$ is Hermitian with respect to the inner product (A16), we get for the normalized eigenvectors $\psi_{\varkappa,n}$ the following orthonormality conditions

$$\left(\psi_{\varkappa,n}, \psi_{\varkappa',n'}\right) = \varkappa\delta_{\varkappa,\varkappa'}\delta_{n,n'} \, . \tag{A27}$$

The set $\psi_{\varkappa,n}$ is complete in the space of two columns dependent on $\mathbf{x}$. An explicit form of the completeness relation may be written if one takes into account equations (A17) and (A27),

$$\sum_n \left[\psi_{+,n}(\mathbf{x})\overline{\psi}_{+,\mathbf{n}}(\mathbf{y}) - \psi_{-,\mathbf{n}}(\mathbf{x})\overline{\psi}_{-,\mathbf{n}}(\mathbf{y})\right] = \delta(\mathbf{x} - \mathbf{y}) \, . \tag{A28}$$

One can easily see that the equation (A22) retains his form under the following substitution $\epsilon \rightarrow -\epsilon, \; q \rightarrow -q, \; \varphi \rightarrow \varphi^*$. That means that that the energy spectrum ϵ^c of the Klein-Gordon equation for the charge $-q$ is related to the energy spectrum ϵ of the Klein-Gordon equation for the charge q by the relation $\epsilon^c = -\epsilon$.

Using similar consideration, it is possible to present a solution of the eigenvalue problem for the charge conjugated Hamiltonian $\hat{h}^c$, see Eq. (A19). In fact, the result may be derived from (A25) by use (A19). It reads (see Fig. 1):

$$\hat{h}^c\psi^c_{\varkappa,n} = \epsilon^c_{\varkappa,n}\psi^c_{\varkappa,n}\ ,\ \psi^c_{\varkappa,n} = -\sigma_3\psi^*_{-\varkappa,n}\ ,\ \epsilon^c_{\varkappa,n} = -\epsilon_{-\varkappa,n},\ \left(\psi^c_{k,n}\ ,\psi^c_{\varkappa',n'}\right) = \varkappa\delta_{\varkappa,\varkappa'}\delta_{n,n'}\ . \quad \text{(A29)}$$

It is easy to see that the charge conjugated solutions $\psi^c_{\varkappa,n}$ obey the same orthonormality conditions (A27) and the completeness relation (A28). The latter being written in terms of ψ and ψ^c takes the form

$$\sum_n \left[\psi_{+,n}(\mathbf{x})\overline{\psi}_{+,\mathbf{n}}(\mathbf{y}) + \sigma_{\mathbf{3}}\psi^{\mathbf{c}*}_{+,\mathbf{n}}(\mathbf{x})\overline{\psi^{\mathbf{c}}_{+,\mathbf{n}}}^*(\mathbf{y})\sigma_{\mathbf{3}}\right] = \delta(\mathbf{x}-\mathbf{y})\ . \quad \text{(A30)}$$

It involves now only positive energy solutions for particles and antiparticles.

Time dependent set of solutions $\psi_{\varkappa,n}(x)$ of the Klein-Gordon equation (A12), which is related to the stationary set of eigenvectors $\psi_{\varkappa,n}(\mathbf{x})$, reads as follows:

$$\psi_{\varkappa,n}(x) = \exp\{-i\varepsilon_{\varkappa,n}x^0\}\psi_{\varkappa,n}(\mathbf{x}). \quad \text{(A31)}$$

It is complete and obeys the orthonormality conditions (A27).

4. Quantized scalar field

In course of the quantization (second quantization) the fields φ and Π become Heisenberg operators with equal-time commutation relations $[\hat{\varphi}(x), \hat{\Pi}(y)]_{x^0=y^0} = i\delta(\mathbf{x}-\mathbf{y})$, which imply the following commutation relations for the Heisenberg operators $\hat{\psi}(x)$ (operator columns of the form (A11)) and $\hat{\psi}^c = -(\hat{\psi}^+\sigma_3)^T$:

$$[\hat{\psi}(x), \hat{\overline{\psi}}(y)]_{x^0=y^0} = [\hat{\psi}^c(x), \hat{\overline{\psi}}^c(y)]_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y})\ . \quad \text{(A32)}$$

Equations of motion for the operators $\hat{\psi}$ and $\hat{\psi}^c$ have the form

$$i\partial_0\hat{\psi}(x) = [\hat{\psi}(x), \hat{H}^{FT}(x^0)] = \hat{h}(x^0)\hat{\psi}(x)\ ,\ i\partial_0\hat{\psi}^c(x) = \hat{h}^c(x^0)\hat{\psi}^c(x)\ , \quad \text{(A33)}$$

where $\hat{h}(x^0)$ and $\hat{h}^c(x^0)$ are defined by (A13) and (A19) respectively. The first equation (A33) implies the Klein-Gordon equation (A2) for the Heisenberg field $\hat{\varphi}(x)$.

In external backgrounds, which do not create particles from the vacuum, one may define subspaces (in the Hilbert space of the quantum theory of a field) with definite numbers of particles invariant under the evolution [28–32]. An important example of the above backgrounds are nonsingular time independent external backgrounds[9]. Let us consider below such kind of backgrounds to simplify the demonstration. A generalization to arbitrary backgrounds, in which the vacuum remains stable, looks similar [32].

One may decompose the Heisenberg operator $\hat{\psi}(x)$ in the complete set (A31),

$$\hat{\psi}(x) = \sum_n \left[a_n \psi_{+,n}(x) + b_n^+ \psi_{-,n}(x) \right] \ . \tag{A34}$$

It follows from the commutation relations (A32) and from the orthonormality relations (A27) that $[a_n, a_m^+] = [b_n, b_m^+] = \delta_{nm}, \ [a_n, a_m] = [b_n, b_m] = 0$. Thus, we get two sets of annihilation and creation operators a_n, a_n^+ and b_n, b_n^+, which may be interpreted as ones of particles with a charge q and antiparticles with a charge $-q$. Indeed, the quantum Hamiltonian and the operator of the charge, which may be constructed from the expressions (A14), have the following diagonal form in terms of such creation and annihilation operators

$$\begin{aligned} &\hat{H}^{FT} = \hat{H}_R^{FT} + E_0, \ \hat{H}_R^{FT} = \sum_n \left[\epsilon_{+,n} a_n^+ a_n - \epsilon_{-,n} b_n^+ b_n \right] = \sum_n \left[\epsilon_{+,n} a_n^+ a_n + \epsilon_{+,n}^c b_n^+ b_n \right] \ , \\ &\hat{Q}^{FT} = q \sum_n \left[a_n^+ a_n - b_n^+ b_n \right] \ , \end{aligned} \tag{A35}$$

where $E_0 = -\sum_n \epsilon_{-,n} = \sum_n \epsilon_{+,n}^c$ is an infinite constant, and $\hat{H}_R^{FT}$ is a renormalized Hamiltonian, namely the latter is selected as the energy operator.

The Hilbert space $\mathcal{R}^{FT}$ of the quantum field theory may be constructed in the backgrounds under consideration as a Fock space. One defines the vacuum state $|0>$ as a zero vector for all the annihilation operators $a_n|0>= b_n|0>= 0$. The energy of such defined

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[9]As examples of singular time independent external backgrounds one may mention supercritical Coulomb fields, and electric fields in time independent gauges with infinitely growing potentials on the space infinity

vacuum is zero. A complete basis may be constructed by means of the action of the creation operators on the vacuum, $a^+_{n_1}\dots a^+_{n_A} b^+_{\alpha_1}\dots b^+_{\alpha_B}|0>, \quad A, B = 0, 1, \ldots$. At a fixed A and B the basis vectors describe states with A particles and B antiparticles with given quantum numbers respectively. A state vector of the quantum field theory in a given time instant x^0 we denote as $|\mathbf{\Psi}(x^0)>$. It evolutes with the time x^0 according to the Schrödinger equation with the renormalized Hamiltonian $\hat{H}^{FT}_R$,

$$i\partial_0|\mathbf{\Psi}(x^0)>=\hat{H}^{FT}_R|\mathbf{\Psi}(x^0)> \ . \tag{A36}$$

In the time independent background under consideration each subspace $\mathcal{R}^{FT}_{AB}$ of state vectors with the given number of particles A and antiparticles B is invariant under the time evolution, since the Hamiltonian $\hat{H}_{FT}$ commutes with number of particles operator $\hat{N}$,

$$\hat{N}=\sum_n\left[a^+_n a_n + b^+_n b_n\right] \ . \tag{A37}$$

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